Making Math

Developing Student Mathematicians

Computer-Aided Manufacturing (CAM) Teacher Notes

These notes sketch possible directions for your classes as they apply their CAD skills to projects that produce real objects using available 3-D printing or laser-cutting equipment.

If you have not already done so, share some of the videos listed in the Teacher Notes in the Virtual Version of a Real World Object Project file. They convey some of the breadth of applications for CAM.

Students can be given specific engineering challenges, or they can be given prompts that allow for wider explorations and possible applications. Just as an art teacher doesn't welcome their students to class and say "do some art", engineering/STEM teachers need to provide structured challenges (we can't just say, "invent something"). But, as an art teacher may provide methods, materials to work with, themes, constraints (e.g., limited time for quick sketches), models, or styles to study, engineering projects can be framed in a number of different ways.

Project 1 Options for 3-D Printers: One can start design projects with a relatively focused task. Some possibilities:

- Students who have been working with Lego robotics might be asked to **design a new Lego-style piece** that would have been helpful and that they could print out. Personally, I always wanted bricks with bumps on their sides to facilitate right angled turns.
- They could be tasked with **designing a fixture** (shelf, lighting, hook, etc.) **or decoration for their school locker**. It could clip on, hang from, or otherwise attach to their locker without glue.
- They could work with a science or art teacher to **design a needed tool for holding or storing items**.
- Develop a design that is readily scalable/adjustable for an object that is frequently in a grouping such a Russian doll, a set of figurines in a family or crowd, measuring

cups, telescoping objects, paintbrushes, leaves, seashells, stop motion character frames, etc.

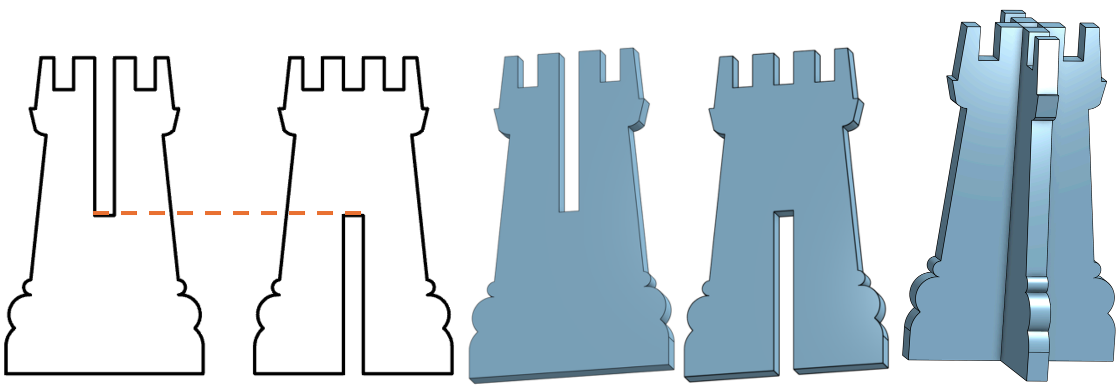
Project 1 Options for Laser Cutters: If you or your students have access to a laser cutter, **press fit and slip fit designs** are great techniques for projects. These designs focus on combining the flat forms that laser cutters cut to make fuller 3-dimensional objects. Possible project themes include:

laser-templates.com/products/3d-elephant

A Class Menagerie – Each student picks an animal and designs pieces that slide and tab together so that the animal's main body and appendages (legs, wings, antennas, etc.) are all in place. See the elephant at right, many examples here, and the Building a Dinosaur video (apologies to the creators – I have not been able to source this clip). Once a student has the main components of their animal, they can do additional work adding details and textures to the shapes that are etched or cut all of the way through.

- **A Miniature Furnished Room** – You can share with students pictures from The Thorne Miniature Rooms at the Art Institute of Chicago. These tiny, but exquisitely detailed, rooms were built by craftspeople to capture a particular architectural and artistic period and a specific room function (e.g., kitchen, living room, etc.). If the class picks a period that is less ornate (art deco, contemporary), they could each choose an item such as a chair, table, couch, bookcase, the frame of the room, etc. to produce. The scale for the room has to be agreed upon in advance so the pieces all work together (as they would in a doll house). Laser cutters can cut and etch leather and fabrics, so even upholstered pieces could be attempted.

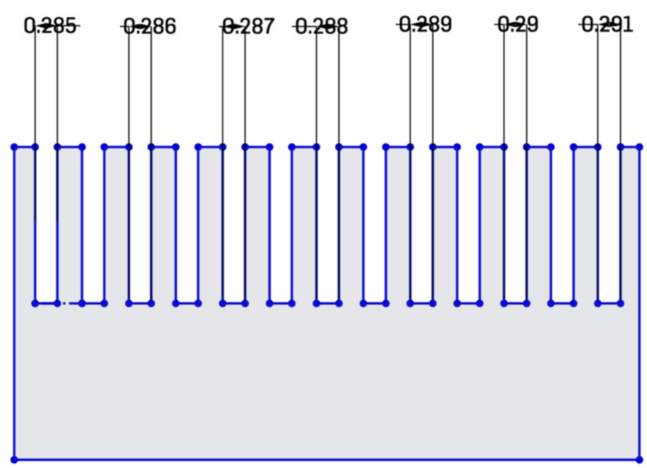
Slip fit designs can be easier than press fit designs, because they are more forgiving of tolerances in fit. Slip fit designs involve sliding two or more pieces with complimentary gaps together as shown with the model for a chess rook below. The first two shapes are sketches in OnShape. The next two are those sketches extruded. The final is an OnShape assembly showing how they would look when cut and put together.



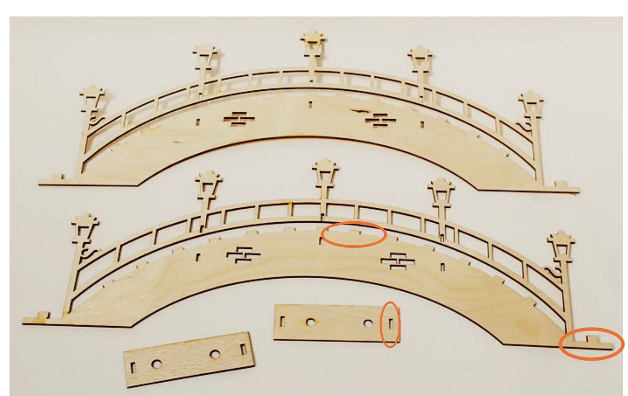
For a slip fit design, to make sure the tops and bottoms of the pieces meet, the starts and ends of the groves should line up as shown with the dashed orange line above. Where they line up is not critical. The top groove could be a little shorter so long as the bottom one is longer. Their total length should equal the height of the pieces being combined.

Complicating slip and press fit designs is that laser cutting removes material (the cut is not infinitely thin) and so physical measurements won't exactly match those dimensioned in the CAD draft. Laser cutters work by burning away/vaporizing a thin portion of the material. The width of missing material is called the **kerf** of the cut. Depending on the power and speed of the laser cut and the material being cut, the kerf can vary. We want the resulting width of the grooves to match the thickness of the material being cut (which should be a global variable, *thickness*, in any related OnShape sketches).

Slip fit designs hold together with more or less wiggle depending on how snug the fit is, but there is some leeway to the fit without a noticeable difference in how they stand. To get a feel for potential fit, students can make a slip fit tester. They should first measure the width of the material they plan to use with a caliper. For example, I have a piece of wood I want to use for the rook that measures 2.88 mm in thickness. Depending on where I did the measurement, the width actually ranged from 2.85 to 2.91 mm. The tester below, which was drafted in OnShape, has rectangular gaps dimensioned 2.85 mm, 2.855 mm, 2.86 mm, 2.865 mm, etc. through 2.91 mm. When I laser cut the tester out of my chosen material, the width of each rectangular cut-out will be affected by the dimensioned width and the kerf. I can see what those measurements produce in terms of actual width by just sliding a piece of the material width-wise into each slot. It may not fit the smallest ones at all and is likely to be too loose for the largest, but somewhere in the middle is the size I will want to use and can then set the *width* variable to in OnShape.



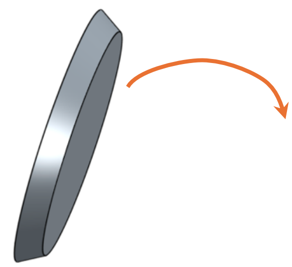
Press fit designs are a little more complicated, but also quite satisfying. The goal is to make a structure that holds together firmly without the use of any glue. They are typically tab and slot constructions such as can be seen in the bridge below (advertised on Etsy). The tabs



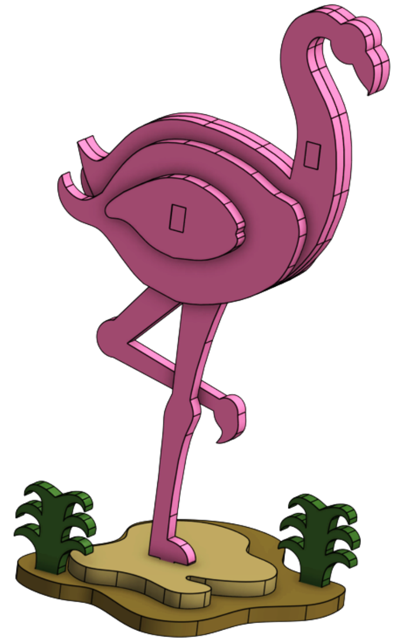
highlighted in the orange ovals for the flat pieces in the image at the right are inserted into matching slots, whose width is precisely determined to make a tight fit.

The fit of a press fit design is aided by a quirk in laser cuts. Unlike the uniform width of a kerf made by a jig saw, the gaps cut by a laser are wider at the top of the cut, where the laser first encounters the material, than at the bottom. This difference is due to the top portion's longer exposure to the heat of the laser as the cut reaches to the bottom of the material. So, rather than have a channel with vertical walls, the sides actually angle inward as shown here with the two lighter brown sides of a cut. The diagram shows a press fit in action. We want the top of the kerf to be wide enough to permit the tab to slide in and the bottom to be a little narrower than the tab, so it grabs hold when the tab is pressed in firmly. Press fit designs are held together by friction. To learn more about kerf, see this video.

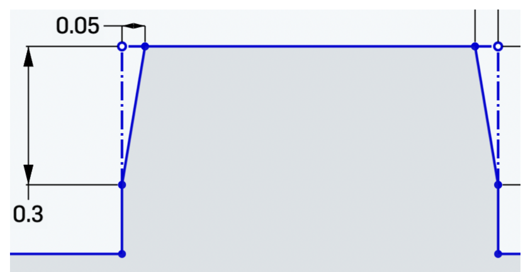
One easy way to show the angled nature of the kerf is to lasercut a circle (which will make a shallow cylinder or coin-shaped chip). If you stand the circle on edge and roll it, it will always roll in an arc (as shown in the exaggerated diagram at right), because the edge is angled and the coin is tilted to one side. The direction of the roll is toward the edge that was the top of the material for the cut.



For the box at left, there is a combination of slip fit connections between the vertical walls and press fit connections between each wall and the floor. The top flap is held in place so long as the side walls can't splay out at all (there are two small circles that tabs on the flap fit into and can rotate within). The headphones shown here were designed by a 7th grader who wanted to make a functional press fit object. He removed the speakers from a cheap pair of headphones and designed the enclosures to lock each one into place. The headphones are not actually comfortable, but they do stay in place and do play music when plugged in. The press fit tolerances were tight enough that, while I brought the headphones with me to countless school fairs over two decades to show them off, they stayed together reliably.

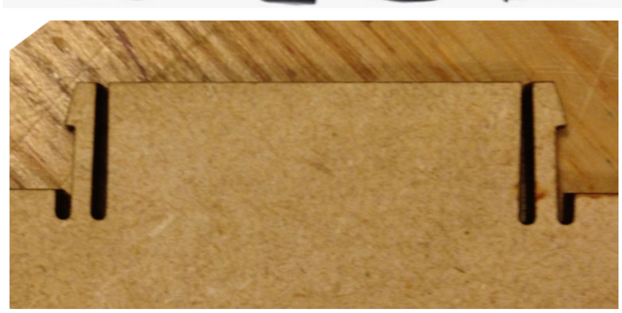


Tip: *Test Designs Before Using Nice Materials* Wood and acrylic can be expensive materials. It is best to save them until iterative testing of a design has been carried out more affordably. The first way to test a model is to extrude the parts in OnShape to the thickness of the material a student plans to use. Once extruded, these parts can be assembled to make a virtual edition of the object. At right is a virtual flamingo for which the student incorporated a setting to facilitate one-legged standing once it was no longer virtual. While in their assembly, students can test for interference to make sure the parts match well. Once a model seems right, they can laser cut a version out of corrugated cardboard as a final check (making the *thickness* variable and the extrusions match the thickness of the cardboard).



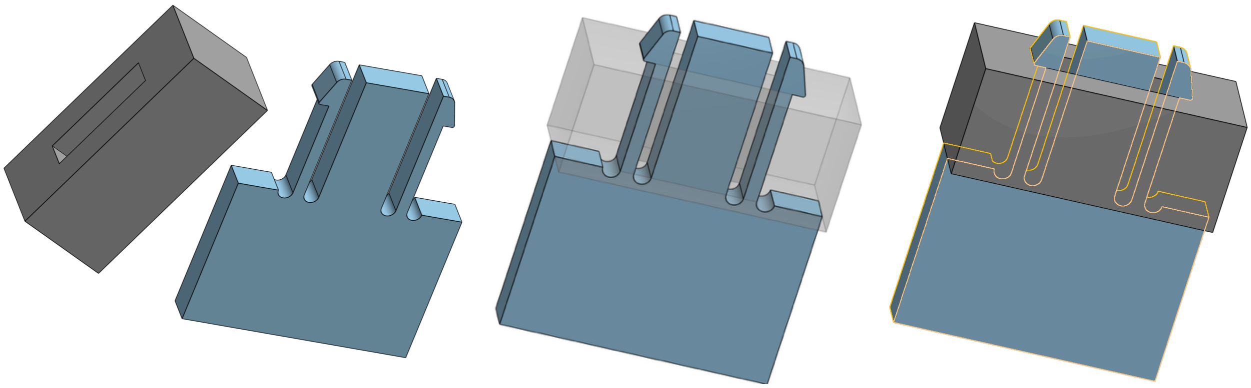
Tip: *Shape Your Tabs* If you fillet or gently chamfer the corners of your tabs, you can create a tight fit in the direction of the breadth of a tab in addition to the width of the material due to kerf. Chamfers can be dimensioned differently for each edge of a corner as shown in the illustration at right. If the width of the hole this tab is going into is a tiny bit less than the width of the base of the tab, but wider than the top, then the tab will fit in readily at first and can then be tapped in with a mallet gently for a strong connection.

Of course, there will be times when you want your structure not to shift of slide apart under any circumstances. While you may want to initially ban glue so that students have to attend more carefully to issues of precision, glue is not against the law! Nor are bolts, which help hold together the Loopita (https://www.left.mx/index.php/products/loopita), an amazing slip-fit lounge chair for two.



Tip: *Snap Fit Joints Hold Firm* There are many ways to join wood or other materials together. One that is related to a standard tab and slot arrangement is a snap fit joint. With tabs, we typically size them so that they are flush with the far side of the slot. With snap fit joints, which are similar to the buckles on backpacks and luggage, the tab has two flexible prongs on either side that pass all the way through the slot and hold firmly on the other side. The tabs have to be able to bend enough to squeeze through and then return to their original position.

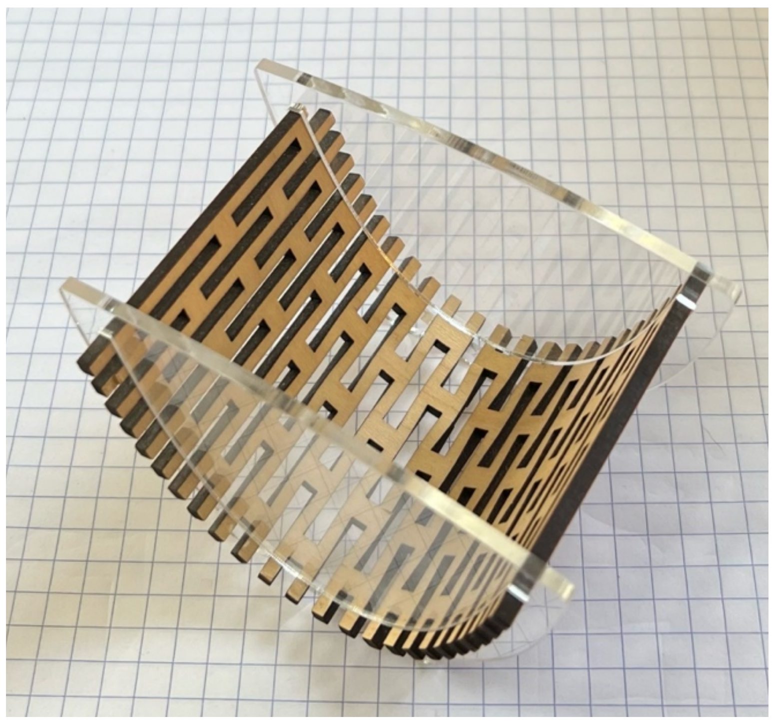
Below are models of the two parts involved. The gray block could be part of a wider piece of material. Its depth should exactly match the distance between the flat base near the prongs and the lip of the top of the prongs (as shown in the two transparent views). The rounded troughs on either side of the prongs reduce the stress caused when the prongs bend inward. Squared off cuts make edges that are more prone to start a fracture in a material being asked to bend. The rounded areas distribute those stresses more evenly.



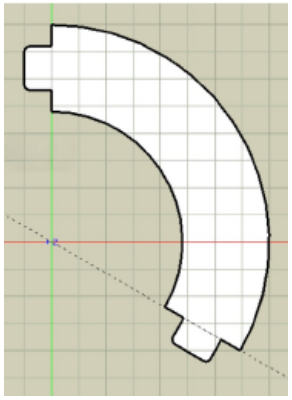
Snap fit joints can be used in countless ways. In the design at right, the joint and its slender bendable arc are the whole component adding handles to a drawer. While the material the handles came from was not very flexible, by thinning the structure, it gains some degree of flexibility. You can experiment with different wood and acrylic boards to see how much objects can bend without breaking.

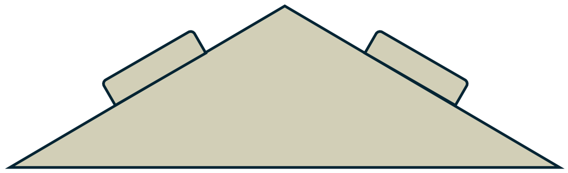
Tip: *Flat Materials Don't Have to Stay Flat*

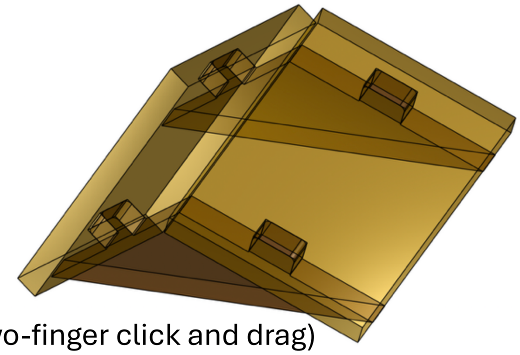
Parts do not have to be small and thin to become flexible. They can be big overall, but locally thin. Living hinges (like the base of the business card holder below) take advantage of this idea in an extreme way – a good portion of the material is removed throughout the object leaving many connected slender regions. This pattern is flexible in only one direction, but some patterns allow a surface to bend along their width and height. Living hinges are surprising, eye catching, and very time intensive as they need the laser to cut away so much of the original material. Do an internet search for "living hinges" and you will see a remarkable variety of aesthetically pleasing patterns and useful applications.



Tip: *Planar Components Don't Have to Meet Perpendicularly* The simplest tab and slot design makes two flat forms that meet at right angles. With the addition of a third component that serves as an internal or external support, such as the arc and triangle below, forms can meet at any angle needed.





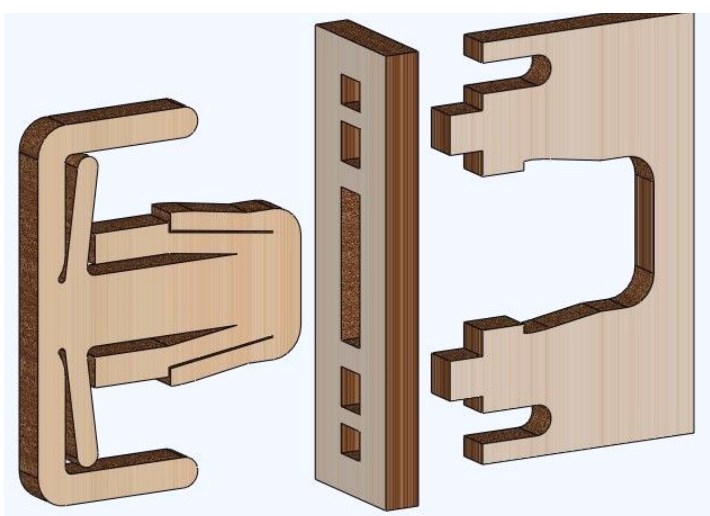


Click here to rotate this example (with a two-finger click and drag)

Darkly Labs

Slip fit designs can be as elaborate as one's imagination and patience permit. The lamp fixture here shows how thinking about cross sections – mostly parallel for the base and radial for the lamp – can help build up a sense of true volume from planar components.

Tip: *Search the Web for Inspiration* As mentioned earlier, ways to join parts of a structure are endless and people have been endlessly creative coming up with new approaches. For example, if you search for "Japanese joinery", you will see many beautiful ways to connect components without nails. If you are 3-D printing, some of these might be helpful for making larger structures than can be 3-D printed by your printer (e.g., if you want an object that is bigger than 10" of 12" or want to avoid too much wasted support materials). Here are two parting examples: an incredible snap fit joint and a way to connect shapes that are coplanar that could have been a single shape, but which were too big for the laser cutter.



www.pinterest.com/pin/130534089179304924

Tip: *Using Onshape with a Laser Cutter*

Laser cutters typically take .SVG files as input. If students are working in OnShape, they can create sketches, extrude them, and then export drawings based on those sketches. See the video "Exporting a File for Laser Cutting" on our page of CAD videos for a demonstration of the steps to produce an .SVG file.

Tip: *Keeping Track of Print/Laser Jobs*

OnShape (and CAD files in general) include the measurements and scale of the models that are exported. Despite that, many 3D printers and laser cutters use software that does not properly import the design at its proper scale. Alternatively, a student may want to change the scale of an object because, while they should use variables to control scale within OnShape, more elaborate shapes with curves don't always rescale well even in a well-dimensioned design with global variables. So, scaling the whole object from an .stl file may be the simpler approach. Once a file is imported into the interface for a 3-D printer or laser cutter, there is the opportunity to rescale it and students need to keep track of the scale factors they use, or different parts printed at different times will no longer work together. So, have a clipboard with a Printing (or Cutting) Log by the machine which all students use to record their jobs and any scaling they do to their imported files. Sometimes, there is a trial-and-error process to get to a size that seems right, so the Scaling Steps keep track of each tweak and the Total Scale is the product of those scalings.

Printing Log					
Date	**Name**	**Project**	**Filename**	**Scaling steps**	**Total Scale**
5/28/26	Josh	Conic Section Model	parabola.stl	0.5, 0.8	0.4
5/28/26	Josh	Conic Section Model	ellipse.stl	0.5, 0.8	0.4
5/28/26	Luanne	Farm Mobile	barn.stl	150	150
/ /					
/ /					
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Project 2: One way to inspire students to come up with new ideas and to provide focus is to provide specific objects and have students brainstorm ways to incorporate one (or more) of them into a design for a device or artwork that they would like to make. Good objects for incorporation have these features:

- They have fixed geometry that students need to measure accurately to incorporate or build around.
- They pose real constraints (size, strength, ways to attach, etc.) and opportunities.
- They have simple forms (mostly circular, square, etc. – they are not *flowy*).
- They can be used in different, open-ended ways.

Here is a list of some of my favorites. I would typically pick 3 of these for students to consider.

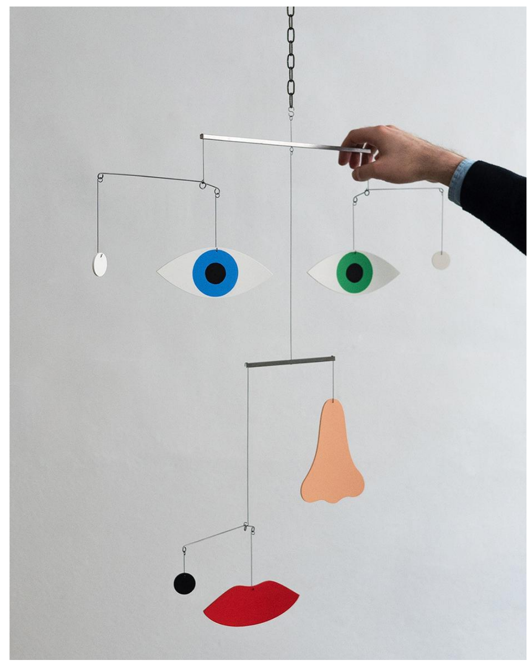
- Suction cups with attachment knobs
- Toy submarine or boat motor

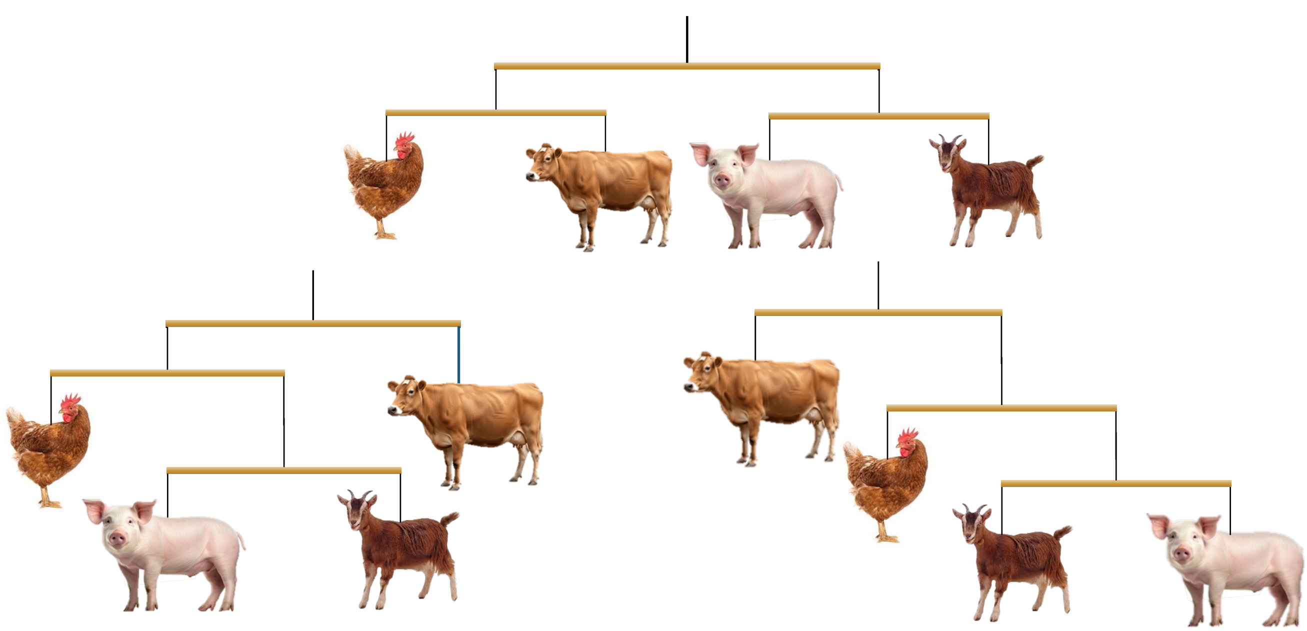
- Wooden dowels (small and medium)
- Narrow plastic piping
- Ball bearings
- Round magnets
- Springs
- Craft sticks with holes
- Small lightbulbs (here and here)

Project 3: Calder's Mobiles. In the early 1930s, building on the work of Man Ray and others who were experimenting with kinetic art, Alexander Calder invented the mobile – a kinetic sculpture of balanced hanging objects. There is a lot of great geometry and physics used in the design of mobiles that CAD programs can facilitate.

Here are stages for designing a mobile:

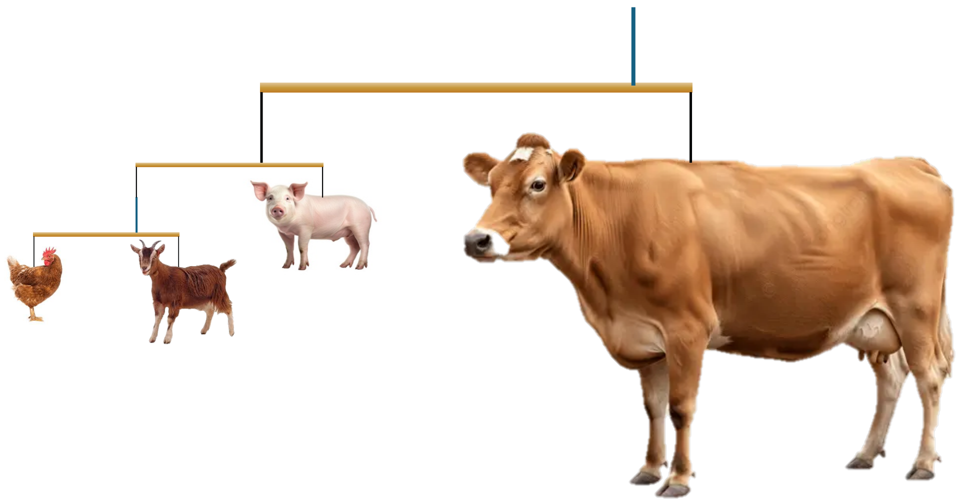
1. Pick a theme for the mobile. This can be a playful or serious topic or a favorite hobby or interest of the student.
2. Choose the shapes that fit the theme and how they should be arrayed in the mobile. Placement may matter, as with the face mobile by illustrator Francisco Ciccolella, or it may not connect to the theme in a relevant way because there may not be a natural hierarchy to the forms. Shown below are two ways – even and staggered – to arrange a cow, pig, goat, and rooster. Note that the two staggered examples are actually equivalent (mobiles can move about!).


francescociccolella.com/mobiles



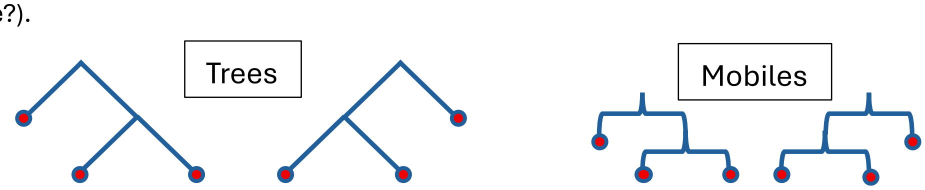
The number of possible configurations grows as the number of elements in a mobile grows and students can experiment with the different aesthetic results.*

The animals above are roughly equal-sized. A more realistic and challenging mobile would involve 3-D printed animals or laser cut silhouettes that are accurately scaled to the real animals. With this approach, the balance point for the groupings would have to accommodate the different weights with the attachment point closer to the heavier component as shown here (see torque activities below).



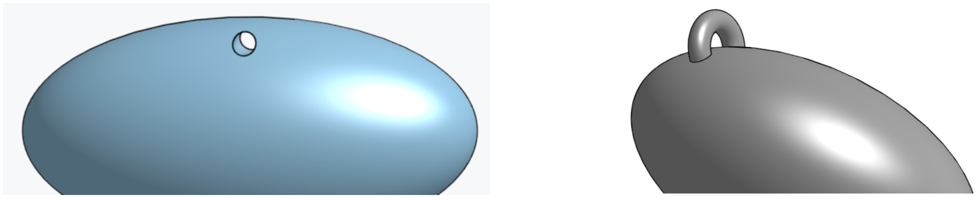
3. If the pieces are to be 3-D printed, they can be designed with three-dimensional forms. If they are to be laser cut, they can be flat. Students can complete preliminary designs for their first two or three components in OnShape, but they are not yet ready to incorporate a hole or loop for attaching the shape until they decide how they want the figure oriented and find its center of mass (see COM activities below).
4. A mobile should be built incrementally so students can refine their understandings and techniques as they go and work toward their desired artistic vision gradually. Have students begin with two shapes that are the bottom of their portion of the mobile: there is a just a connecting bar and an object on each side such as the goat and chicken above and not another connecting bar below. Before students work in a CAD program further, teach them about how to orient individual hanging shapes and

* This is an interesting mathematics question related to the question of how many distinct binary trees exist with *n* identical leaves, but the mobile version has a much smaller answer. For example, the two trees with 3 leaves at the left are considered distinct, but the two mobiles are free to move and rotate around any hanging string, so they are equivalent. Students might want to explore this as a math research problem (what notation or representations might help describe the mobiles? How can one compare two mobiles and check for equivalence?).



how to balance a bar. Both of these tasks require an understanding of torque and an object's *center of mass* (COM). See below for explanations and activities related to the physics and application of torque and center of mass.

5. Once the class has had their physics and geometry investigations, they can return to their first shapes and introduce a hole (with an extrude remove) or add a loop (by

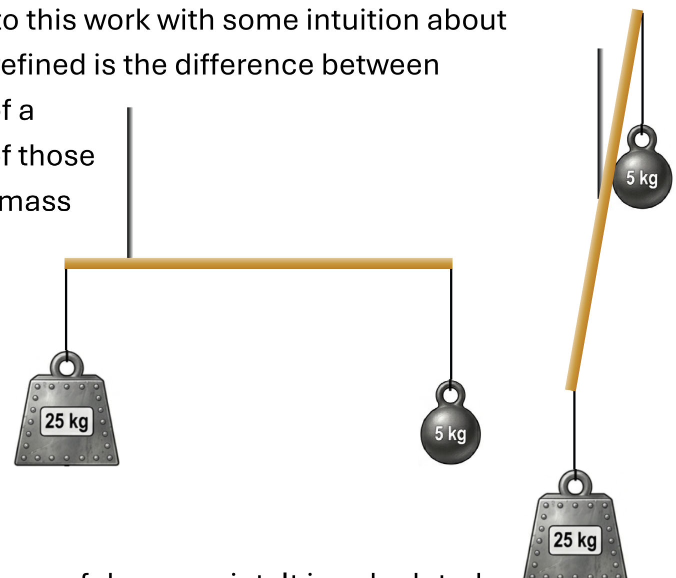


revolving a circle around a segment outside of the circle) near the boundary of the figure.

6. Once a student has two objects hanging from a connecting bar and the bar hanging balanced from a string, they are ready to work their way *upwards* as they complete more portions of their mobile. For example, for the mobile on the prior page, after making a balanced goat and chicken, they would make the CAD design for their pig and balance it against the goat-chicken combo.

Torque Activities: What Is Torque and How Do We Measure It to Balance Our Mobiles?

We want to be able to hang shapes so that they are oriented the way we want and also so that they balance each other. Students will come to this work with some intuition about balance. The main idea that typically needs to be refined is the difference between balancing the amount of weight on the two sides of a fulcrum (balance point) and between the *impact* of those weights. It is perfectly possible to balance a 25 kg mass with a 5 kg mass, so long as we place them in the correct places. At right, we see a failed balance between a 5 kg and a 25 kg mass hung from a wire attached to the midpoint of their beam. No one will be surprised that this doesn't work. The imbalance is due to the 25 kg mass exerting more torque on the beam than the 5 kg mass.



Torque is caused by a force exerted at a distance from a fulcrum point. It is calculated by multiplying force*distance. What we are comparing in the balances above is torque. With the beam on the right, the 25 kg mass is exerting 5 times as much torque as the 5 kg mass because they are the same distance from the middle. For the balanced situation on the left, the smaller mass is five times farther from the fulcrum point from where the beam is hanging (e.g., 150 cm versus 30 cm), and so the torques are equal and the beam balances.

Physics Note: *Mass* is a measure of how much material is in an object (or of an object's resistance to being accelerated). Metric measures such as kilogram reflect mass.

Measures like pounds are *weights* from the British imperial system – they reflect the *force* exerted by a mass under the influence of gravity. Mass does not change regardless of its setting, but weight does (people and things weigh less on the moon, but their mass is the same). When we say 1 kg = 2.2 kg, we are associating kg masses with the force in pounds they exert on Earth. A 1 kg mass is still 1 kg on the moon, but it weighs only 0.37 pounds. The metric measure of force (and therefore of weight) is Newtons (in kilogram-meters per second squared).

Unfortunately, I assume because they led to too many accidents, see-saws are less common in playgrounds than they once were. I am glad small children are not getting bopped by see-saws, but I am sad that they are missing out on the many physics experiences see-saws provided. As children of different sizes worked to keep a see-saw balanced, they discovered that the bigger kids had to shift closer to the fulcrum point. They learned that that placement reduced how much rotational force they exerted on the see-saw. In the picture, the larger child weighs twice as much as the smaller child and so moved halfway toward the fulcrum to balance out the see-saw. If the girl is Weight*Distance, then the boy is (2*Weight)*(1/2*Distance), and those two expressions are equal.

Elementary and middle school students are sometimes taught about different types of *simple machines*, such as levers, pulleys, and inclined planes – which are ways to produce mechanical advantage so that large objects can be moved will less force. See-saws and mobiles use lever principles, which is why a smaller weight can balance a larger one.

Torque Demonstrations

Door Challenge: This first demonstration can be done with some flair. Make sure everyone in the class can see the classroom door. It is often more entertaining if you pick a student who thinks they are strong or tends to banter with you. Ask them to place their hands on the classroom door (even closer to the hinge than shown here, but not so close that there is a risk of finger-squashing) and to try to push you out of the class. You stand by the outer edge. On the count of three, they should push with all of their might. You should be able, with both nonchalance and fanfare, to push the door and push them backwards. Depending on where they are pushing, you may be able to do it with just one hand or a finger or two. Ask the students why you were able to do this. Ask for other examples from life that show the benefit of being farther out from a hinge. You can note that someone using a wrench to turn a bolt does not hold the handle close to the bolt but as far out on the handle as possible (and that longer handled wrenches give them even more leverage or torque to turn a tight bolt). If you don't want this push version of a tug-of-war, you can also have students just open and close a door using just one or two fingers first near the handle and then by pushing an inch from the hinge. They will feel the difference in force needed to achieve the same rotation of the door. This equates to a heavier weight close to the fulcrum being equivalent to a lighter weight farther away.

Improvised See-saw: You can also make a simple adjustable see-saw with an 8' x 10" x 2" wooden plank and a 4" x 4" x 10" (or longer) piece of wood (these sizes are not the only ones that can work). Students can then experiment with putting the smaller fulcrum piece of wood at different locations and having one student stand at an end and seeing how another student can balance them or, to avoid discussing students' weights, you can make it more qualitative by having students push on one end and see where they have to put the student standing on the plank to be able to lift them. Note that the wooden plank itself has significant weight and can't be fully ignored in any calculations unless the fulcrum is the middle and the weight on each side cancels out.

Desktop See-saw: Scaled versions of the above setting can be done with meter sticks and smaller masses. See the video *Table-top see-saw demo* on our CAD videos page. Standard masses (e.g., 100 mg cylinders) for experiments are expensive when purchased from science companies, but you can get a common unit of mass to work with by buying a box of large washers, hex nuts, or hex bolts from a hardware store.

Mini-mobile: When giving students hand-on experiences, it is helpful to develop their physical intuitive understandings before returning to a quantitative approach. Provide students with a small dowel (8-12"), or chop stick or straw (not the bendy flexible type), and some strands of string or yarn, and have them tie or tape two different objects of an appropriate scale (not very heavy) to either end of their cross beam. They can then tie a third piece of string somewhere between the ends of the cross beam at a place that they think will balance the system. They can temporarily tape it in place, if it is slipping. If the location they choose does not work (the beam is not basically horizontal), they can slide the knot for the third string until they find the balance point. They can experiment with different weights either by hanging more items from one end (e.g., using a wire hook and washers as shown in the *Mini-mobile materials and steps demos* video) or by attaching some malleable material such as Playdoh to the ends of the beam. After students have a balanced system, they can measure the distances and the masses (a metric ruler and an inexpensive digital kitchen scale that shows grams is sufficient equipment) and see how closely $\text{weight}_1 * \text{distance}_1 = \text{weight}_2 * \text{distance}_2$.

Materials List (for both this activity and the main mobile project)

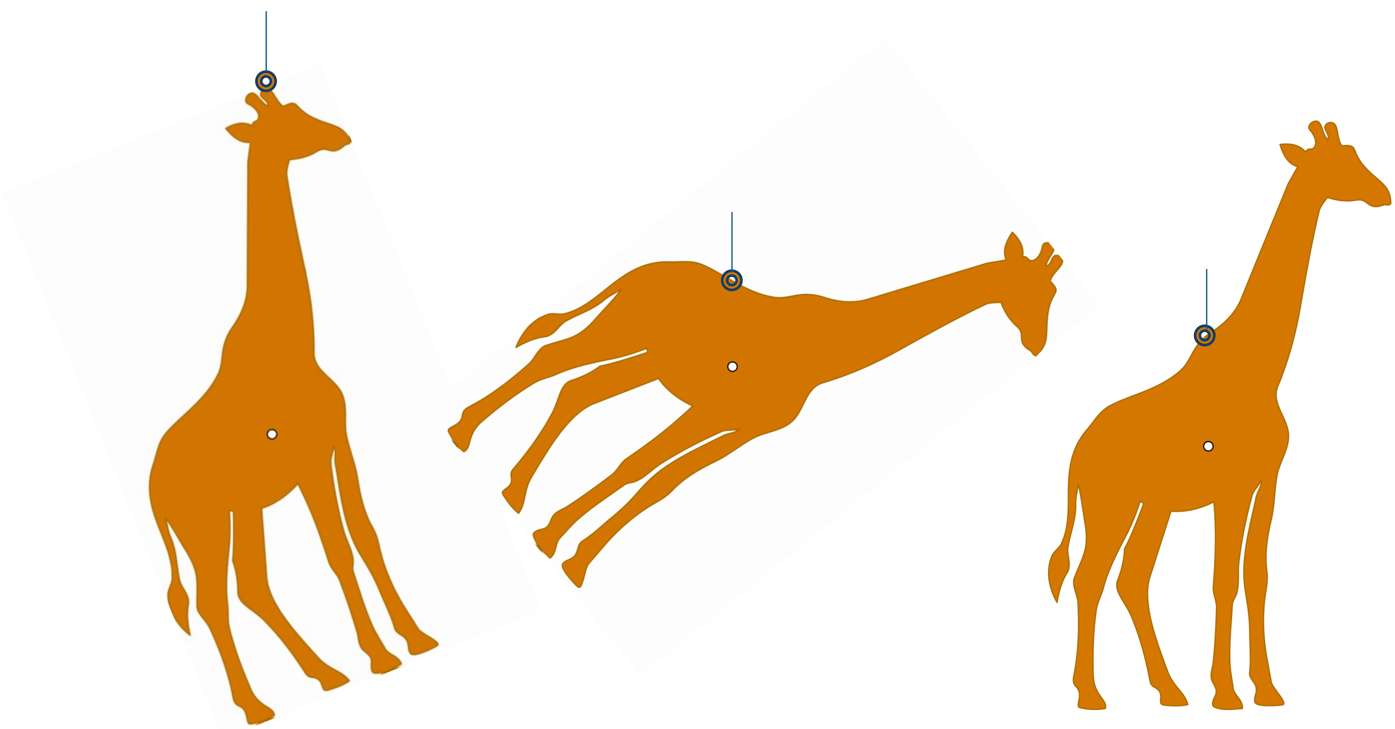
- String or yarn
- Scissors

- Tape
- Either 1/8" wooden dowels (or 3/16", if students are building a mobile with heavier objects on longer beams) or 18 gauge galvanized wire (or 16 gauge, if students are building a mobile with heavier objects)
- Needle-nosed cutting pliers (can cut both the wire and the 1/8" dowels).
- Optional: Curtain clips (e.g., these) as shown in the *Mini-mobile demo video*.

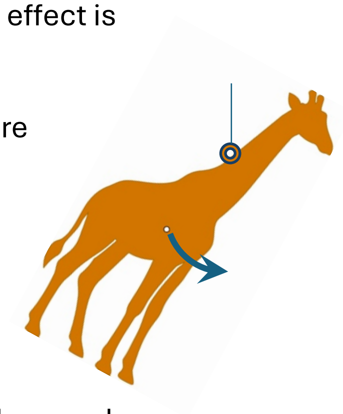
Just What Is the Center of Mass?

The design of a mobile not only requires balance between the different components, but also for each item itself. If an object has symmetry, it is usually clear where on the object you would want to attach its string. See, for example, the different components of the face mobile shown previously. All are attached at the top of the vertical line that defines their horizontal symmetry. But, if a shape is not symmetrical, or you want a specific orientation that does not use that symmetry, then understanding the concept of center of mass is important.

In the example below, a student is hanging a giraffe for an animal-themed mobile. Initially, they attach the string at the highest point of the giraffe (as shown at left), but it does not hang with the feet all level as hoped for. They then try moving the attachment point to the back, but the weight of the head, extended far out as it is for a giraffe, produces too much torque, and the giraffe turns too much in the clockwise direction. In the attempt on the right, we see the point that leaves the giraffe standing upright.



Why is the attachment point on the right the correct location? The white dot in the middle of each giraffe is its center of mass – this is the point where all of the mass and how that mass is distributed balances out the torques. You can see that in all three cases the giraffe ends up in a position where *the center of mass is directly below the attachment point*. The center of mass is the single point where you can pretend all the mass of an object is concentrated. While gravity pulls on every part of an object, the combined effect is the same as if all of the mass were at the center of mass.



We can simplify our analyses by pretending all of the gravitational forces are one large force acting at the center of mass. If that point is not directly below the point from which an object is hanging, the object will rotate so that the center of mass is at its lowest possible point (because gravity pulls things down) and that will be right below its attachment point.

Physics side note: you can demonstrate the center of mass idea by flipping the above discussion. If you take an object, such as a tall glass or box, and start tipping it, it will not fall until its center of mass is no longer above the base of support. If the center of mass is above the base area, torque will pull it back upright. That is a stable arrangement. If it is outside, torque will make it fall (it is in an unstable position). An object tips when its center of mass moves outside its base of support. This is why it is easy to tip over a marker that you have balanced on its end. Only a slight perturbation moves the center past the narrow base and the marker will fall. In contrast, it is hard to poke at a tissue box to make it fall over. It has to rotate about 45° to flip to a new side, whereas the marker only has to tilt a very few degrees. A water bottle is between these two extreme cases.

Building up our intuitions about center of mass

The Hands-on 1-dimension video demonstration (https://www.makingmath.org/cad-videos) shows fun and quick activities for finding the center of mass balance point for objects that are long and thin (their mass is essentially distributed along one axis or dimension). You can do an attention-getting demonstration with a bat or broom and then have students try out the process with smaller lab sets of dowels and Playdoh.

Materials List

- Meter sticks
- Small pieces of wood or other suitable “fulcrum boxes”
- Standard weights (washers, bolts, etc.) and other random objects to test
- Playdoh
- 1/8” dowels
- Optional: Baseball or whiffle ball bat. Baseball bat donut or packing tape ring.

How to hang a shape the way you want

So, if, as shown with the giraffe above, we can hang a shape the way we want by attaching the string to a point above the center of mass, how do we find the center of mass, especially for irregular shapes? For shapes that are essentially flat (such as laser cut items or extruded forms), the shape has a plane of symmetry halfway between the two congruent surfaces. The center of mass will be on that plane. To find its location, we can focus on where in two-dimensions it lands for one of the main faces. We can find out manually using plumb lines or automatically in OnShape. Here are the two methods.

Cardboard cutouts. These steps are demonstrated in the *Center of Mass: For Cardboard Cutouts and Other Flat Objects* video.

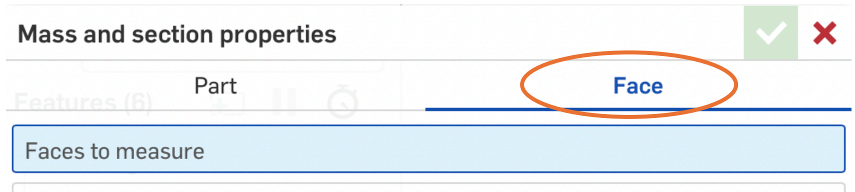
1. Have each student cut an asymmetric shape from cardboard (or poster board). Ask them to approximate and mark where they think its center is.
2. Using string, thread, or yarn, attach a weight (e.g., a washer or bolt, not something asymmetric itself that might skew the measurement) on one end and tie it to a push pin with the other end.
3. They should push the pin in along the edge anywhere on the shape and spin and angle the pin so that there is not much friction when it turns. If they hold only the pin and the shape can spin readily and hang freely, that is the goal.
4. The weight and string constitute a plumb bob, which shows which direction is straight down relative to gravity. When the cardboard rotates on the pin, the center of mass will rotate right below the pin. The sting gives us a sightline for marking where the center might be.
5. Students (this is easier with a partner) can pinch the string onto the cardboard where it is hanging, place it on a table and make a mark with a pen. They can now put the plumb bob aside for a moment.
6. They should then use a straight edge/ruler to draw a segment that connects the hole where the pin was to their pen mark. The center of mass lies somewhere on this segment, because that is the set of all points directly below the pin.
7. They should then repeat this process from two other points along the edge of the shape. All three segments should pass through the same point or very close to the same location. That point of intersection is the center of mass.
8. If their point is accurate, they should be able to balance the shape on a finger placed under that point.

An object's center of mass does not have to be within the object. For example, a cutout of the letter C (or a donut) will have a center of mass in empty space.

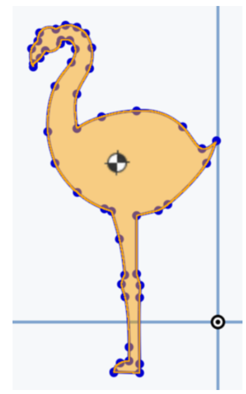
Using OnShape. These steps are demonstrated in the *Center of Mass: Using OnShape for 2-D Objects* video.

1. The sketch in OnShape should be an enclosed region. If it is, its interior will be light gray.
2. While still in a sketch, click on the balance icon in the lower right corner.

3. That brings up the Mass and section properties window.
   a. Click on the Face tab.
   b. Click in the interior of the shape whose center of mass you want.



   c. A circular black and white target will appear where the center of mass is. Unfortunately, there is no way, yet, in OnShape, to automatically add that location as a point in your sketch. So, look closely for some landmarks, and then leave the properties window and use the point tool to add a point there.



   d. Once you have the point in place, open the properties window to make sure the target covers it up. If not, adjust and repeat.
   e. The attachment point should be above the point you just placed, so use the line tool to draw a segment straight up from the point and add a small circle close to the edge of your shape along that segment. This will lead to your laser cutter or 3-D printer making a hole right where you want to attach your shape.

So, now you know how to find the center of mass for shapes that vary mostly in one- or two-dimensions, but what about normal three-dimensional forms? After doing the above activities, students should have a good feel for where the center of mass is for any objects that they plan to 3-D print. Some CAD programs, including OnShape, will show the point, but that location may not exactly match the center of mass for a printed object, because 3-D printing will typically print the surface of a shape with thick layers and then fill in the hollow interior with thin scaffolding. That approach means that the distribution of mass in the printed shape is not the same as for a solid model. So, some approximation for attachment points may be needed.

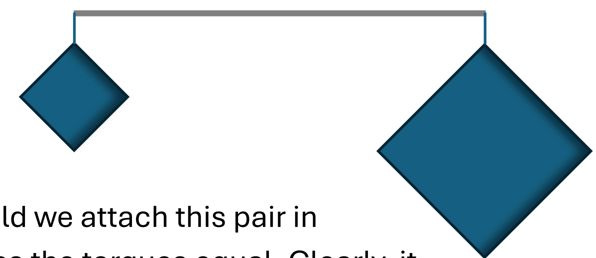
The center of mass of a 3-D object can be found by hanging the object from two or more different points and seeing how it hangs. As with the cardboard cutouts, where the plumb lines meet is the center, but those lines and the center are probably buried within the figure, so one has to visualize them as best as possible.

Getting the balance right: algebra to the rescue!

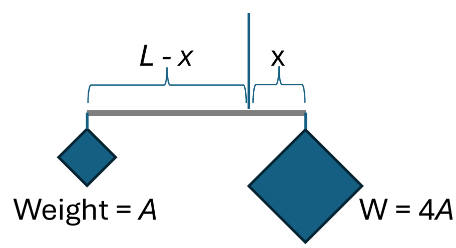
As we have seen, the center of gravity is important for two reasons: One, objects can be hung from a point that is above the center of gravity and that keeps them oriented the way the artist wants. Two, the same idea is relevant to balancing a system of objects, such as two hung from a cross beam. When two objects hanging from a rod stay balanced then the attachment point must be above the whole system's center of mass.

Center of mass is about mass and how far it is distributed from the center – it is about equal torque. It is not about finding the point with half of the mass on each side. If a shape is not symmetrical, there will probably be more mass on one side than the other, but that mass will be closer to the center of mass.

To find the balancing point for where to hang a pair of objects, the masses on the two sides don't have to be equal, but the attachment point from which their beam is hung has to create equal torques (rotational forces). If you look at the red mouth and the black beauty mark in the face mobile above, you will see that the attachment point is very close to the heavier mouth. The weights are not equal on the two sides, but the torque (weight*distance from the balance point) is.



For the mini-mobile here, with two squares of equal depth and material, the square on the right has four times the area, so its weight, *W*, is four times as much as the smaller square's. Where on the connecting bar (with length *L*) should we attach this pair in order to balance it? We want the point on the bar that makes the torques equal. Clearly, it has to be much closer to the big square than the small one. But where? Let's do some algebra. Let's use *x* to represent the distance from the right end of the beam to the attachment point as shown in the diagram.



Torque of big square = Torque of small square

Weight * Distance = Weight * Distance

$$4A \cdot x = A \cdot (L - x)$$

$$4Ax = AL - Ax$$

$$5Ax = AL$$

$$x = {}^{L}/_{5}$$

So, the attachment point should be a fifth of the way along the bar from the large square. This makes sense! Since $x = \frac{1}{5}L$, the remaining portion, $L - x$, is equal to $x = \frac{4}{5}L$. So, the lengths have a one to four ratio and are in inverse proportion to the weights. That is always going to be the case. The attachment point will divide the bar in the same ratio as the weights but with the bigger length for the smaller weight. When multiplied, the ratios cancel out and we get the same torque. I encourage you to have students work solving similar equations, first for a specific case of objects in their mini-mobiles and then the most general case.

Specific Case from the Mini-mobile Video	The Most General Case
1939 NY World's Fair mirror – 16.5 grams Empire State Building coaster – 24 grams Length of cross beam – 17.5 cm x – the distance between the attachment point and the mirror $16.5x = 24(17.5 - x)$ {set torques equal} $16.5x = 420 - 24x$ $40.5x = 420$ $x = 10.4$ cm Check: the attachment point is closer to the heavier object. Good!	First Object – M_1 Second Object – M_2 Length of cross beam – L x – the distance between the attachment point and the mirror $M_1x = M_2(L - x)$ $M_1x = M_2L - M_2x$ $M_1x + M_2x = M_2L$ $(M_1 + M_2)\,x = M_2L$ $x = \dfrac{M_2L}{M_1 + M_2}$ Check: if M_2 is the heavier mass, the fraction is greater than half of L, so the lighter mass (M_1) is farther from the attachment point. Good!

Note: all of these calculations ignore the mass of the cross beam. For the wire with lighter objects or the dowels with heavier objects, that simplifying assumption works out reasonably well. In the see-saw video, I did not explore putting the fulcrum off-center, since the weight of the meter stick was significant in comparison to the washers. Having more of the ruler on one side would have definitely skewed the results – its contribution to torque on each side would not cancel out.

Project Notes:

- There is a project assignment document posted. Depending on the grade you are teaching and how much students have worked on the physics and math aspects of mobiles, you may want to adjust the requirements.
- Students should work the bottom of their mobile up. That way, the mobile always works and can be considered completed even if they run out of time for a more ambitious design.
- Students can predict the attachment point if they know how much an object is going to weigh. As mentioned above, OnShape can't know how an object will be 3-D printed and so it can't know the mass of the finished piece. However, for laser cut shapes, it can provide this information:
  1. In OnShape, have students extrude their designs to the thickness of the material that they are cutting (which will often be 1/8" acrylic).
  2. Then have them right-click or two-finger click on the part itself or the part's name in the Features list on the left.

3. From the menu, choose “Assign Material for *Part 1*” (or whatever the name of the part is). There is a built-in list of metals, plastics, woods, and more. Acrylic is one option.
4. Once a material is assigned, clicking on the Display mass and section properties button will list the objects mass when made from the chosen materials. This requires that the original sketches were completed at the desired scale (which students should be instructed to do as they work).

5. Once they have the predicted mass for the two objects, they can carry out the algebra steps above to see where the attachment point should go. Know this in advance enables students to understand the true aesthetics of their design in advance and to make and desired changes ahead of time. For example, they may not want every beam to be held from the middle. Calder often made shapes cascade with attachment points close together.
6. David Taylor’s video on centers of mass (https://www.youtube.com/watch?v=qRsJXXb9WNE) shows general ideas discussed in this document as well as a really surprising and fun physics connection spinning an irregular object (from 3:20-4:35 of the video).
7. This discussion of whippletrees, which are sort of a sideways mobile, is an interesting connection between mobiles and agriculture.

Project 4: Portfolios

There are many activities and projects in the *01: Shape, Dimension, and Measurement* and *02: Computer-Aided Design (CAD) - Putting Geometry Into Action* units. You can let students know from the very start that they will be assembling a portfolio of their best work and that it should sample from drawings, designs, and a selection of CAD efforts that demonstrate a range of the skills that they will learn (e.g., a use of constraints, of variables, of assemblies, of lofts, etc.). I leave it to you to come up with the criteria and specifications, but having a collection at the end that reflects their growth with geometry can be quite impressive.