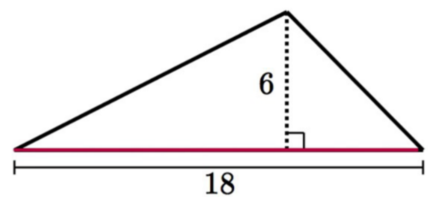
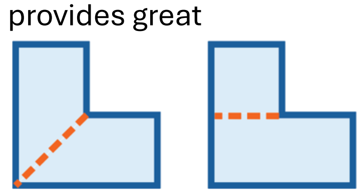
Making Math

Developing Student Mathematicians

Make Measurement Hands-On

When teaching about perimeter, area, surface area, volume, and their units, it is important to provide students with frequent hands-on experiences measuring real two-dimensional and three-dimensional objects. If one looks at textbooks, handouts, or online activities about measurement, students are typically presented with shapes oriented in the most familiar way (e.g., the base of a triangle is horizontal and below the rest of the shape), with all needed lengths labeled, and with no unneeded lengths provided. See a typical example of this kind of exercise at right. This kind of work does not develop understanding or skills or demonstrate that a student can apply either in unfamiliar situations. The introduction to this unit includes a Materials List on the second page that suggests a range of objects students can use to practice surface area and volume measurements. The Area Exercises Collection (**pdf** **doc**) consists of a rather random assortment of problems whose chief characteristic is that they differ from the above example in some important way. Many include no measurements and students need to measure any useful lengths themselves. All students should have metric rulers in class and at home that they can use. As they work, you will discover some students who don't readily visualize where a triangle's altitude (height) is. They may just measure a side or not draw the height perpendicular to the base. I have taught students who struggle to picture and draw right angles or who do not know how to align a ruler correctly. This more varied work helps diagnose and remediate such gaps. The exercise collection is available in Word document form (which you can also open in a GoogleDoc) so that you can skim the collection and use problems that meet your needs for a handout or other assessment. It is not intended as a coherent handout itself. In addition to ruler, pencil, and paper activities, work with random physical objects provides great opportunities to help students organize their work and learn how to deal with complex shapes that are pieces and parts of more familiar ones. Do they see a ring as a circle within a circle or an L as two rectangles or two trapezoids?





The remainder of this discussion provides background information, much of which is foundational information for students, with activities for students interspersed.

Units

Measurements without units are typically meaningless. If a student provides me with a unitless answer, I will ask them to ask me my height. When they do, I just make up number like 53. I point out that they don't know if my answer is correct or not, because they don't know 53 of *what*. Students tend to see units, at best, as something one tacks on after finishing calculations, rather than as a guide to those calculations. See the Making Math Numeracy Estimation and Units materials for ways to help students see how units inform our calculations.

Not only can units give meaning to measurements, they help us understand perimeter, area, and volume formulas. Note these examples (for which constants lack a unit and variables are units of length):

> Perimeter of a rectangle = 2W + 2L. A unitless 2 times a length is just a different length and then a length (2W) plus a length(2L) is still a length, so our answer is still just a one-dimensional perimeter.
>
> Area of a rectangle is WH or length times length, which is length^2, a unit of area. Even when a formula looks more complicated, it still will land at the right dimension: a formula for the area of a trapezoid is $\frac{1}{2}(base_1+base_2)h$, or (unitless constant)(length+length)length which simplifies to (length)length or length^2.
>
> Here is a messy one. The volume of a truncated cone with height h and radii for the two bases R and r is $\frac{1}{3}\pi h(R^2 + Rr + r^2)$. $\frac{1}{3}$ and π are unitless, h is a length, and each term in the parentheses is quadratic (squared) and add up to a quadratic term. So length times quadratic is cubic, or volume.

In each of the above examples, we see that the dimension for what is being measured naturally arises from the structure of the formula and the variables involved. This is an important concept that shows up throughout mathematics and physics. Ask students to perform this kind of an analysis on the formulas and it will help them both understand and remember them.

In the United States, we are forced, just for practical everyday reasons, to teach customary units (e.g., feet, miles, gallons, etc.), but we should cover these are quickly as possible and focus on metric measure, which is consistent (every conversion factor is a power of 10), almost universally adopted in the world, and required for all work in science. Introduce students to the most common metric units (deci 10^{-1}, centi 10^{-2}, milli 10^{-3}, micro 10^{-6}, and nano 10^{-9} on the small end and kilo 10^3, mega 10^6, giga 10^9, and tera 10^{12} on the high end). There are exercises and activities that expand students' skills with metric measure in the Numeracy: Understanding Small and Large Quantities unit.

To help students understand how units build dimensionally and how to compare different units, you can do the Personal Units activity and then the Trimetric Units handout that are in this 2-D shapes sequence. Both of these activities are about understanding the conceptual basis of units free from anything that needs memorizing. Any unit of length can be the basis for length, area, and volume, and any unit can be the basis for a whole range of units when supplemented with a consistent set of prefixes.

Shapes

Before heading into work with CAD software, some geometry vocabulary should be reviewed (or introduced). When presenting definitions, be sure to emphasize their *inclusive* nature. For example, students often come out of elementary school thinking a rectangle has to have different length sides. That is, there are squares and there are rectangles, and the two sets are distinct. But their definitions make it clear that all squares are also rectangles. Similarly, all equilateral triangles are also isosceles and all parallelograms are trapezoids. These inclusive relationships make it easier to find patterns and state properties without having to incorporate unnecessary exceptions. Below are the main definitions for two-dimensional figures.

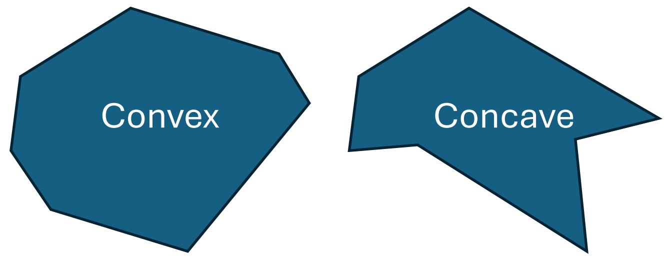
A **polygon** is a closed, planar (2-D) figure made up of line segments connected at their endpoints.

Quadrilaterals (*quad* means four, *lateral* means side) are four-sided polygons.

A **triangle** is a three-sided polygon.

Other polygons are **pentagons** (5 sides), **hexagons** (6), **heptagons** (7), **octagons** (8), **nonagons** (9), **decagons** (10), and **dodecagons** (12).

Polygons can be **convex** (each segment connecting any two points in the shape also lies in the shape) or **concave**, which is the same as non-convex (has a dent or "cave").



There are many triangle types, including:

- **Right** – A quadrilateral is a square if all four sides are congruent and all four angles are congruent.
- **Isosceles** – A triangle with two or more congruent sides.
- **Equilateral** – A triangle (or any polygon) where all sides are congruent.
- **Scalene** – All sides have different lengths.
- **Equiangular** – Any polygon with all angles congruent.
- **Acute** – A triangle is acute if all three angles are acute (less than 90°).
- **Obtuse** – A triangle is obtuse if one of its angles is obtuse (more than 90°).

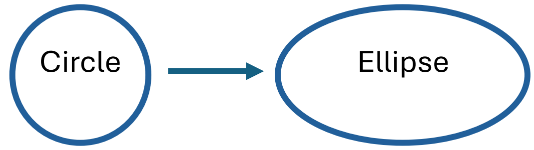
There are many types of special quadrilaterals (ones with special properties). Familiar ones include:

- **Square** – A quadrilateral is a square if all four sides are congruent and all four angles are congruent (it is equilateral and equiangular).
- **Parallelogram** – A quadrilateral is a parallelogram if both pairs of opposite sides are parallel.
- **Rectangle** – A quadrilateral is a rectangle if all four angles are congruent.
- **Rhombus** – A quadrilateral is a rhombus if all four sides are congruent.
- **Trapezoid** – A quadrilateral is a trapezoid if *at least* one pair of opposite sides is parallel.
- **Isosceles Trapezoid** – A quadrilateral is an isosceles trapezoid if at least one pair of opposite sides is parallel and the base angles (two adjacent angles sharing one of the parallel sides) are congruent.
- **Kite** – A quadrilateral is a kite if it consists of two distinct (non-overlapping) pairs of adjacent congruent sides.

Moving on from polygons, we have:

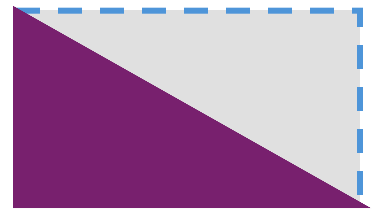
- **Circle** – A circle is the set of all points equidistant from a center point. That equidistance is the length of the radius. The center of the circle is not a point of the circle. The circle is just the circumference. A circle plus its interior is called a **disc**.

- **Ellipse** – Ellipses have many interesting geometric definitions. The simplest for now is a circle that has been scaled non-uniformly (stretched in one direction).

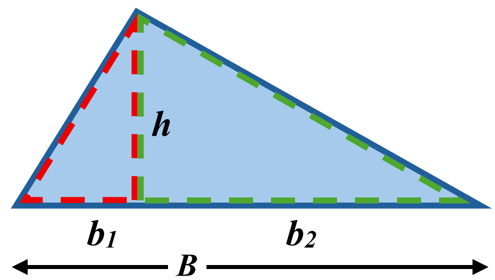


Area Formulas Derived by Dissection and Rearrangement

Because a square tiles a region without overlap, it is the simplest and most natural choice for a unit of area. Rows of squares lead to the formula for rectangles: $L \cdot W$. With the rectangle area formula as a foundation, other area formulas follow.

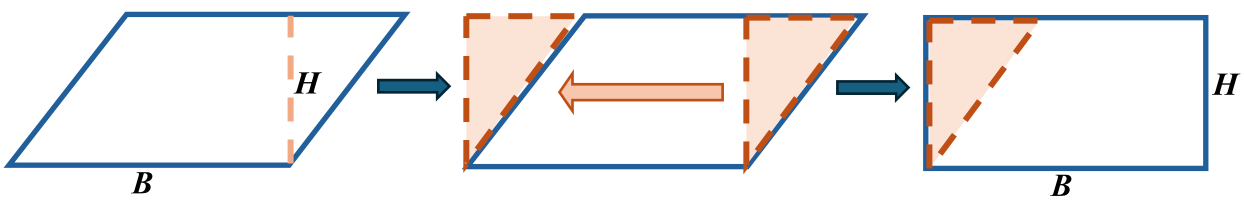


We can construct a rectangle around any right triangle to demonstrate that the area of the triangle is ½ of the enclosing rectangle (image at right). So, we have area of a right triangle = $\frac{1}{2}BH$.

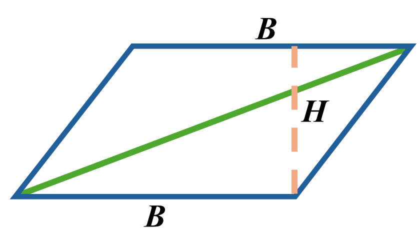


All triangles have at least one internal altitude that divides them into two right triangles. In the diagram at right, the original blue triangle has been subdivided into two right triangles. Using the above formula for right triangles, the total area = $\frac{1}{2}b_1h + \frac{1}{2}b_2h = \frac{1}{2}(b_1 + b_2)h = \frac{1}{2}Bh$. So, all triangles work with the same formula. If we want to show that all three bases can be used to find the area for obtuse triangles, we would need to subtract, rather than add, the areas of two right triangles. See if you can set that proof up.

The formula for parallelograms can be developed in several ways. One is to rearrange the shape to make a rectangle by moving the triangle on one side to the other (a rigorous proof would include showing how everything works out to make the sides straight and the angles right).



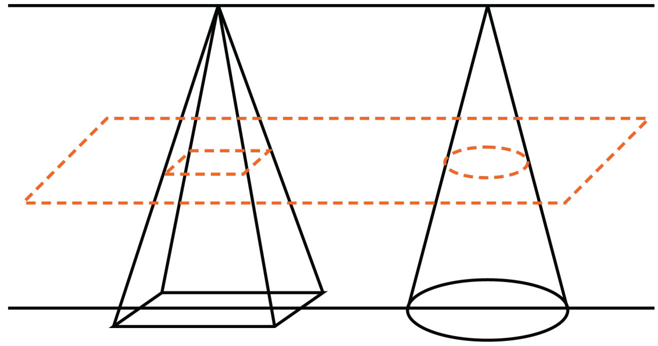
So, the area of the parallelogram is the same as the area of the resulting rectangle which is BH.



The formula for both trapezoids and parallelograms can be found with a simple dissection into two triangles using a diagonal. This approach yields area = $2(\frac{1}{2}BH)$ or BH. For a trapezoid $\frac{1}{2}B_1H + \frac{1}{2}B_2H = \frac{1}{2}(B_1+B_2)H$. This dissection approach requires less additional justification than for the rearrangement proofs.

Cavalieri's Principle

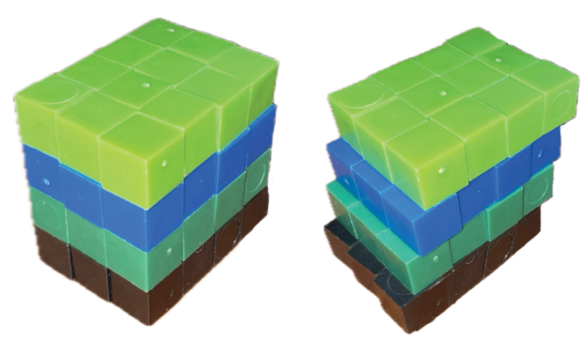
While area formulas are often demonstrated in classes as described above. Thinking about cross-sections is a powerful way to compare areas and volumes that emphasizes the relationship between dimensions. This approach is fruitful in working with CAD software and facilitates understanding of the key operations of calculus: integration and differentiation.



We will start with a volume example of Cavalieri's Principle: Begin with two solids bounded by a pair of parallel planes (shown as bounding lines above and below here). Consider each plane parallel to those two planes (picture it sweeping from the bottom to the top as if it were scanning both solids) and the areas of the cross sections of those solids in that plane. If the area is the same for all cross-section pairs in each plane, then the the two solids have the same volume. At right is an image that is part of a classic derivation of the formula for the volume of a cylinder (the pyramid is already known to be 1/3(area of Base)(Height) which leads to the same formula for cylinders). The proportions of the shapes are chosen so that at each level, the square cross section and circle cross section have the same area, so the two solids have the same volume.

Hands-on work with objects can help students understand this theorem. In the the first picture below, we see a cylinder made of 14 pennies (we are old enough to still remember pennies). Think of each penny as a cross section (even though real cross-sections have no thickness – they are two-dimensional). The other two penny stacks are the same height as the cylinder and each cross section has the same circular area at each height as the first, so they all have the same volume of 14 pennies (with penny as a unit of volume since it has so little worth monetarily) even though the latter two are skewed and, well, wonky. It is clear, however, that they do not all have the same surface areas. The edges of all pennies are exposed in all cases as are the top and bottom circular faces, but in the final two images, we can see additional parts of the faces of the coins increasing the overall surface area. The same would be true if we looked at an oblique (slanted) cylinder. Slanting a figure does stretch and increase its boundary, but leaves the amount of inside (volume) unchanged.



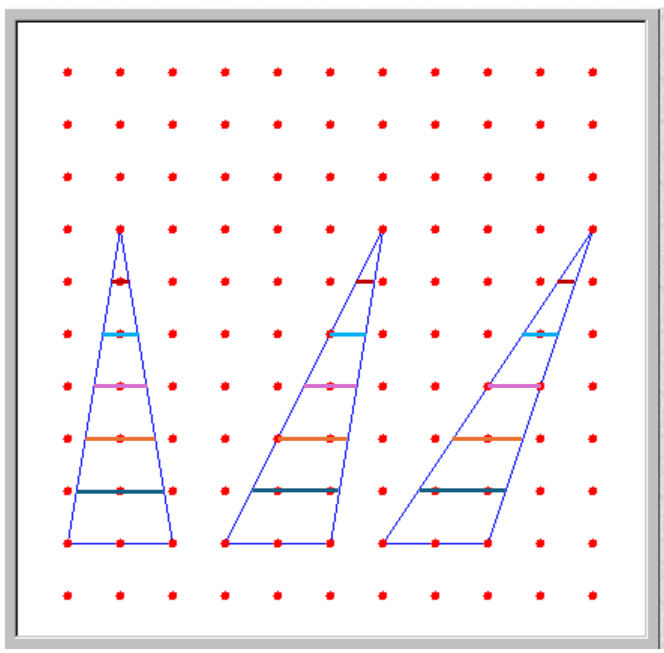


Students can build their own examples with unit cubes. The image on the left is a rectangular solid, or box, with a volume of 3 cm * 4 cm * 4 cm = 48 cm^3. The object to the right has skewed layers, but each horizontal cross section still has the same area as in the box, so the volume is unchanged. The surface area is increased because it has all of the original exposed faces as well as the now-exposed steps on the left and right sides.

Students can be helped to further understand the two-dimensional nature of cross-sections by making shapes (cubes, balls, cylinders) with Playdoh and then using thread or a plastic knife (and a sort of back and forth sawing motion to make clean cuts) to expose internal faces of the shape (see a video demonstration here). Round and square cookie cutters might also work well, but I haven't tried those. The cross section is not the playdoh portion, it is the 2-D surface that has been exposed. Working with cylinders, students can find circular and elliptical cross-sections (as well as portions of ellipses if the planar cut emerges through a base). Cross sections of a tetrahedron (triangular pyramid) can produce different types of triangles (isoceles, equilateral, scalene) as well as a square (can you figure out how?). Cross sections of a cube are many (different types of triangles, quadrilaterals, pentagons, and even a hexagon, but don't tell students about any of these – ask them to experiment and record their discoveries and then share out afterward).

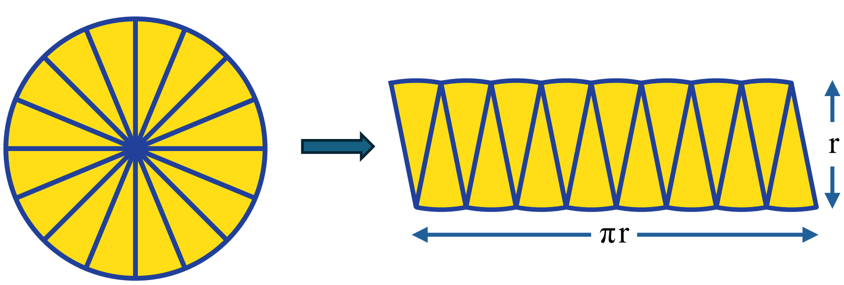
Cavalieri's Principle for Area

In place of dissection, area formulas can be understood through the two-dimensional version of Cavilieri's principle: If two areas are bounded by a pair of parallel lines and each line parallel to these bounds intersects the areas with segments of the same length, then the areas are equal.

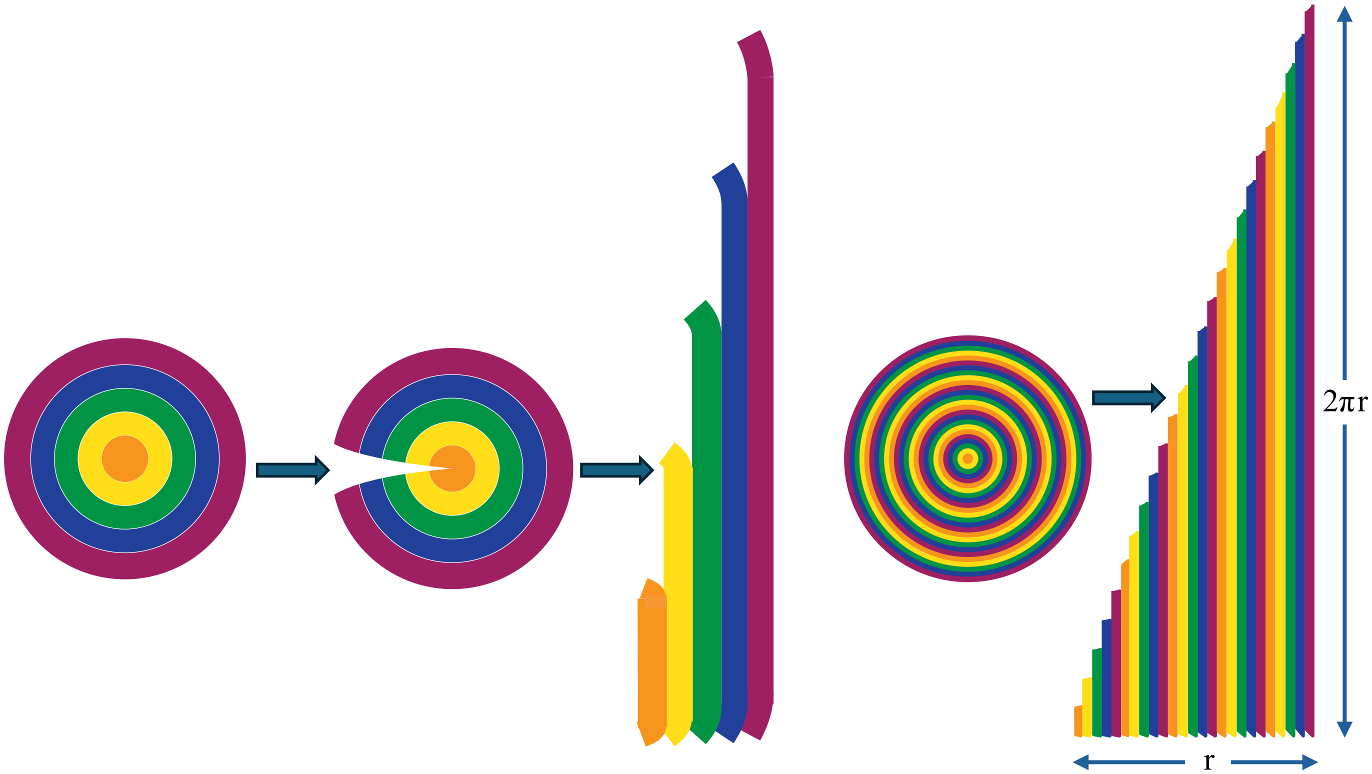


The illustration at right shows how triangles with equal bases and heights have matching cross sections (this could be proven using similarity or coordinate geometry), so any of them could be compared with a right triangle of the same dimensions to verify that $A = ½BH$ for all triangles. Similarly, one can compare the horizontal segments that comprise a rectangle and parallelogram with the same base and height to show that their areas are equal.

There are demonstrations that suggest the formula for the area of circles that involve dissection. The simplest is to cut it into many wedges and then rearrange them into what looks like a slightly lumpy parallelogram. That shape is as high as the wedge's radius and the lumpy tops and bottoms are half of the circumference yielding $A \approx \pi r \cdot r = \pi r^2$. The more wedges we cut the original circle into, the less curvature is in the arc of each wedge and the closer the reassembled shape approximates a parallelogram (or rectangle with the same dimensions).



Another dissection involves dividing a circle into concentric rings, cutting the rings, and unfolding them and, to the extent possible, straightening them. We cannot unbend them into rectangles while preserving their area, because the inside circumference of each ring is shorter than the outside circumference. But, if we use a lot of rings, the unfolded rings better approximate rectangles. If we break the original area into an infinite number of nested circles (each with no



width), then those open up into a triangle made of an infinite number of segments. The triangle has a base of r and a height equal to the outermost disk's circumference, or $2\pi r$. Finding the area of the triangle yields $A = \frac{1}{2}(r)(2\pi r) = \pi r^2$, which is therefore the area of the circle as well. This dissection is Cavalieri-like in the sense that we are pairing equal lengths in the circle and triangle as we sweep radius values from 0 to r.

Optimization

One of the most common objectives in both applied and pure mathematics is optimization: finding the maximum or minimum value for a function, steps needed in an algorithm, etc. CAD is a tool for engineering and design for which optimization is ever-present. Can we make this as well but with less material? Stronger, but not heavier? More aesthetically pleasing while still affordable and effective? In traditional secondary school curricula, optimization typically first appears in calculus, but there are many other mathematical tools for optimization, and even standard calculus fare can be solved for specific cases numerically or graphically well before students know the derivative. See the Making Math Algebra-Equations section for materials on Linear Programming, which utilizes so many introductory algebra skills in powerful and exciting ways, and the Algebra-Functions section for materials on optimizing polynomials and other functions graphically.

In class, give each student 20 to 30 centimeter cubes and ask them to make a rectangular solid (a box with no holes or empty interior) that uses all of them and report out on what dimensions were possible. Students with a prime number of cubes will only be able to line them all up. This question helps students connect the factors of a number with the dimensions possible when that value is a volume.

Next, ask them to make the shape with the smallest possible surface area (no longer limited to rectangular solids). Remind them to count the faces on the bottom that they can't see (but not the faces that are adjacent to other cubes). Have them record their structures and the surface area in a table. After students have had time to explore have them report out on their strategies and observations for limited surface area.

Finally ask them which shape has the greatest possible surface area. In all cases, it should be snake-like with all cubes touching at most two others.

If you have done the above, have students pick up with the second page of the Optimization Activities (pdf doc).