

Calculus in Context

The Five College Calculus Project

Solutions Manual

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Solution Manual for Calculus in Context: The Five College Calculus Project
by James Callahan

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Preface

This *Solutions Manual* consists of detailed solutions to the exercises in 2026 distribution of the textbook *Calculus in Context: The Five College Calculus Project*, by James Callahan and Kenneth Hoffman, with David Cox, Donal O'Shea, Harriet Pollatsek, and Lester Senechal. The original hard-cover book published in 1995 by W.H. Freeman is now out of print.

This *Solutions Manual* is a new edition of an earlier one produced for the *Handbook for Instructors* in 1995. The *Handbook*, without the solutions, continues to be available as part of the 2026 distribution. In this *Manual*, page references to the text are to the 2026 distribution.

Three Smith College students, Lisa Hodsdon, Diane Jamrog, and Marcia Lazo labored over an entire summer in the early 1990s to solve the exercises and to prepare the results as $\text{\LaTeX}$ documents. This *Manual* would not exist without their efforts.

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Solutions: Chapter 1

A Context for Calculus

1.1 The Spread of Disease

Reading a Graph

Most answers here are approximations!

1. The infection hits its peak at about the 13th day, at which time about 14000 people are infected.
2. Initially, about 42000 people are susceptible. The number of susceptibles is halved in about 17 days.
3. The recovered population reaches 25000 in about 33–35 days. Eventually, about 50000 people recover. The information about the total number of those who have recovered is obtained by observing that the number of people recovered given on the R -graph seems to “level off” to the right at about this (vertical) value.
4. The number of people in the infected population seems to be increasing most rapidly during days 6–7. It appears that the I -curve is rising most steeply at about this time. The infected population *decreases* most rapidly around day 19.
5. Since the infected population grows through susceptible persons contracting the disease, the number of people who became infected at some time during the first 20 days should be equal to *number of susceptibles on day 0* minus *number of susceptibles remaining on day 20*; this is about $42000 - 18000$, or 24000.
6. To superimpose the graph of S on the graph of R so that the scales match, you have to compress the graph of S in the horizontal direction.

A Simple Model

7. a. Since $S = 20000 + S't$ and $S' = -470$, after 10 days we have

$$S = 20,000 + (-470)(10) = 20,000 - 4700 = 15300.$$

7. b. To find the number of days it takes for the susceptible population to vanish entirely (i.e., for $S = 0$), we solve $S = 20,000 + (-470)t = 0$ for t and thus find $t = 20000/470 = 42.55$ or about 43 days.

7. c. On the previous Sunday, $t = -3$, so the number of susceptibles then is

$$S = 20000 + (-470)(-3) = 21410.$$

7. d. To calculate the number of days before Wednesday for which there were 30000 susceptibles, we must find t for which $30000 = 20000 + (-470)t$; thus $t = 10000/(-470) = -21.3$ days or about 21 days before Wednesday.

Mark Twain's Mississippi

8. Twain says that L is decreasing at a rate of $1\frac{1}{3}$ miles per year, so $L' = -1\frac{1}{3}$. However, if we *compute* the value of L' using what Twain says is the length of the river in 1708 and 1884, then we find $L' = \Delta L/\Delta t = -242/176 = -1.375$ (or $-1\frac{3}{8}$) miles per year. This is only slightly different from the simpler value of $-1.33\ldots$ that Twain uses. Thus, answers to the following questions will differ, depending on what choice is made for L' .

9. Use the model: new $L = \text{old } L + L'\Delta t$. In 1990, $\Delta t = 106$. If we take $L' = -1\frac{1}{3}$, then $L = 1100 + (-1\frac{1}{3})(106)$ or $L = 958\frac{2}{3}$ miles. (If we take $L' = -1\frac{3}{8}$, then the length is 954.25 miles in 1990.)

10. Using new $L = \text{old } L - 1\frac{1}{3}\Delta t$, with $\Delta t = -742$ and new $L = 1\frac{3}{4}$, we can find

$$\text{old } L = 1\frac{3}{4} - 1\frac{1}{3}(-742) = 991.083133\ldots$$

or $991\frac{1}{12}$ miles. (If $L' = -1.375$, then the river would have been 1022 miles long in 1884.)

11. The change in L in t years = (change in L in one year) $\times$ (t years). Thus the change in L in t years = $(-1\frac{1}{3})t$. (With the alternate value for L' , the change in L in t years = $-1.375t$.)

12. $L = 991\frac{1}{12} - 1\frac{1}{3}t$ miles. (If we use the other value for L' , then $L = 1022 - 1.375t$ miles.)

13. $L = 991\frac{1}{12} - 1\frac{1}{3}(-1,000,000)$ or $L \approx 1,334,324\frac{5}{12}$ miles, so Twain's assertion about the mathematics of the model was correct. (Using the other formula, $L \approx 1,376,022$ miles.)

14. The river was never truly 1,300,000 miles long, nor will it ever be reduced to a mere $1\frac{3}{4}$ miles. The problem in Twain's model arises from assuming that the rate of decrease in the river's length is constant over time—especially, over *very long* times.

The Measles Epidemic

15. If $t = 1$, then

$$\begin{aligned} S' &= -.00001(44446.6)(2903.4) = -1290.5 \\ I' &= .00001(44446.6)(2903.4) - 2903.4/14 = 1083.1 \\ R' &= 2903.4/14 = 207.4 \end{aligned}$$

Thus, when $t = 2$,

$$S = 43156.1 \quad I = 3986.5 \quad R = 2857.4,$$

as in the text on page 12.

16. If $t = 2$, then

$$\begin{aligned} S' &= -.00001(43156.1)(3986.5) = -1720.4 \\ I' &= .00001(43156.1)(3986.5) - 3986.5/14 = 1435.7 \\ R' &= 3986.5/14 = 284.7 \end{aligned}$$

and then

$$S = 41435.7 \quad I = 5422.1 \quad R = 3142.1.$$

17. Using $S' = -953.4$, $I' = 803.4$, and $R' = 150$, with $\Delta t = 2$, we have

$$\Delta S = -1906.8 \quad \Delta I = 1606.8 \quad \Delta R = 300,$$

so

$$S = 43493.2 \quad I = 3706.8 \quad R = 2800.$$

18. a. Using the values of S and I from exercise 17 for $t = 2$, we obtain

$$\begin{aligned} S' &= -.00001(43493.2)(3706.8) = -1612.2 \\ I' &= .00001(43493.2)(3706.8) - 3706.8/14 = 1347.4 \\ R' &= 3706.8/14 = 264.8 \end{aligned}$$

Now using $\Delta t = -2$,

$$\begin{aligned} S &= 43493.2 + (-1612.2)(-2) = 46717.6 \\ I &= 3706.8 + 1347.4(-2) = 1011.9 \\ R &= 2800 + 264.8(-2) = 2270.4 \end{aligned}$$

We see that the differences are $+1317.6$ for S ; -1088.1 for I ; and -229.6 for R .

18. b. At the start ($t = 2$), we have

$$\begin{aligned} S &= 43156.1 & I &= 3986.5 & R &= 2857.4 \\ S' &= -1720.4 & I' &= 1435.7 & R' &= 284.7. \end{aligned}$$

Using $\Delta t = -1$, we get the following at $t = 1$:

$$\begin{aligned} S &= 43156.1 + 1720.4 = 44876.5 \\ I &= 3986.5 - 1435.7 = 2550.8 \\ R &= 2857.4 - 284.7 = 2572.7 \\ S' &= -.00001(44876.5)(2550.8) = -1144.7 \\ I' &= 1144.7 - 2550.8/14 = 962.5 \\ R' &= 2550.8/14 = 182.2. \end{aligned}$$

This leads to the following at $t = 0$:

$$\begin{aligned} S &= 44876.5 + 1144.71 = 46021.2 \\ I &= 2550.8 - 962.5 = 1588.3 \\ R &= 2572.7 - 182.2 = 2390.5. \end{aligned}$$

Now the differences are $+621.2$ for S ; -511.7 for I ; and -109.5 for R .

18. c. The errors are noticeably smaller when $\Delta t = \pm 1$. Thus, the smaller step size produces the better estimates.

19. a. The new transmission coefficient is $a = 0.000005$.

19. b. The new threshold level for S is given by the equation $.000005S - 1/14 = 0$; thus $S = (1/14)/.000005 = 14286$.

19. c. Suppose we start with $S = 45400$. Since this S is *still* above the threshold level (where $S = 14286$), quarantine does not immediately eliminate the epidemic. The number of infected persons will still increase, at least until S is reduced to the threshold level.

19. d. We assume $S = 45400$ at the start, and we want to choose a so that I' is never positive. But

$$I' = a \cdot 45400 - 1/14 \leq 0$$

will be true for any

$$a \leq 1/(14 \times 45400) = .000001573.$$

19. e. The answer here is the largest possible value that a could be given in part (d)—namely $a = 1/14(45400)$.

Other Diseases

20. a. Because $R' = .08I$, the recovery coefficient is $b = .08$ per day, so the recovery period is $1/b = 12.5$ days.

20. b. For the illness to take hold, we must have $I' > 0$. Thus $I' = 0$ gives the threshold value of S . Now $I' = 0$ when $.00002S - .08 = 0$, or $S = 4000$.

20. c. If 100 people are currently ill, then $R' = .08 \times 100 = 8$ people will recover in the next 24 hours.

20. d. If there are 30 new cases during the same 24-hour period, then the susceptible population is decreasing at a rate of 30 persons per day, or $S' = -30$.

20. e. Since $S' = -30$ and $I = 100$, $S' = -aSI$ becomes $-30 = -.00002S \times 100$; hence $S = -30 / -.002 = 15000$.

21. a. From the given information, $b = 1/4$ per day and $a = 0.003 \times \frac{1}{6} = .0005$. Thus,

$$\begin{aligned} S' &= -.0005SI \\ I' &= .0005SI - I/4 \\ R' &= I/4 \end{aligned}$$

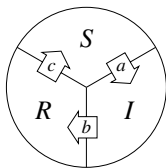
21. b. The illness will fade away without becoming an epidemic if I' is zero or negative at the outset. Setting $I \leq 0$ gives $.0005S - .25 \leq 0$, or $S \leq 500$.

22. a. The threshold level for S is the value of S at which $I' = 0$. This is $S = \frac{b}{a}$.

22. b. If two illnesses have the same a , then the one with the smaller b will have the lower threshold value for S . This is because the threshold value is $S = \frac{b}{a}$ (part (a)). But the length of time someone stays ill is $1/b$ (which gets larger when b gets smaller, and vice versa). Therefore, the illness with the *longer* recovery time $1/b$ will have the lower threshold level for S .

What Goes Around Comes Around

23.



24. Six weeks is 42 days; we convert weeks to days because our basic unit of time measurement is 1 day. Just as we assumed that $1/14$ th of the infected population left I each day, because each person remains in I 14 days, we can assume $1/42$ nd of the recovered population leaves

R each day, since each person remains in R for 42 days. Thus, a reasonable value for c is $c = 1/42$.

25. If this illness lasts 5 days and has a transmission coefficient of .00004 in the population under consideration, and if the population is fixed, we have

$$a = .00004 \quad b = 1/5 \quad c = 1/42.$$

Thus the rate equations are

$$\begin{aligned} S' &= \frac{1}{42}R - .00004SI \\ I' &= .00004SI - \frac{1}{5}I \\ R' &= \frac{1}{5}I - \frac{1}{42}R \end{aligned}$$

26. This model should have a threshold if $I' = 0$. As before, this occurs if $S = \frac{b}{a}$, or $S = \frac{.2}{.00004} = 5000$.

27. a. R will increase if $R' > 0$. Since $I = 2100$ and $R = 2500$, we have

$$R' = .2(2100) - \frac{1}{42}(2500) = +360.5.$$

Thus R will increase.

27. b. If $R = 22500$, and $I = 2100$, then we have $R' = .2(2100) - \frac{1}{42}(22500)$, or $R' = -115.7$. Thus R will decrease.

27. c. There are many possibilities here. For instance, if I remains unchanged at 2100 and we vary R , we have $R' < 0$, if $420 - \frac{R}{42} < 0$, or if $R > 17640$. If R remains unchanged at 2500, and we vary I , we have $R' < 0$, if $\frac{1}{5}I - \frac{2500}{42} < 0$, or if $I < 297.6$.

27. d. Using the same reasoning as in (c), we will have $R' = 0$ if $\frac{1}{5}I - \frac{1}{42}R = 0$, or if $I = \frac{5}{42}R$. Thus, $R' > 0$ if $I > \frac{5}{42}R$ and R will increase, while $R' < 0$ if $I < \frac{5}{42}R$ so that R will decrease in this case.

28. a. If the inflow and outflow to the I compartment are to balance, then $I' = 0$, and $S = 5000$, as in question 26.

28. b. If the inflow and outflow to the R compartment are to balance, then $R' = 0$, and $I = \frac{5}{42}R$.

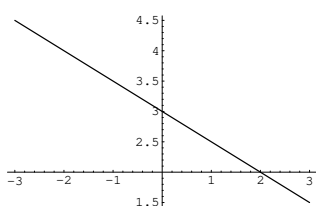
28. c. If neither I nor R is changing, then (since the *total* population doesn't change either), $S = 50000 - I - R$ must be constant. Thus, the model is at the steady state.

28. d. From part (a) we know the value of S at the steady state is 5000.

28. e. $R + I = 50000 - S = 45000$, because $S = 5000$ at the steady state. Because $I = \frac{5}{42}R$ at the steady state, $R + \frac{5}{42}R = 45000$, or $R = \frac{42}{47}(45000) \approx 40213$.

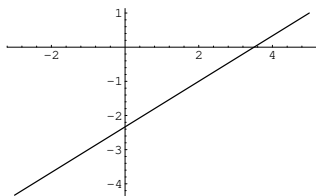
1.2 The Mathematical Ideas

1. a.



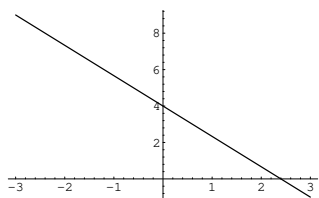
$$\begin{aligned}\text{slope} &= -\frac{1}{2} \\ \text{y-intercept} &= 3 \\ \text{x-intercept} &= 6\end{aligned}$$

1. b.



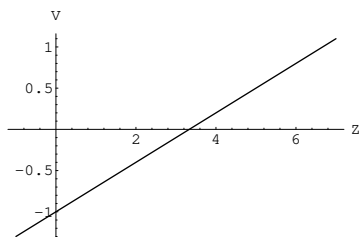
$$\begin{aligned}\text{slope} &= \frac{2}{3} \\ \text{y-int} &= -\frac{7}{3} \\ \text{x-int} &= \frac{7}{2}\end{aligned}$$

1. c.

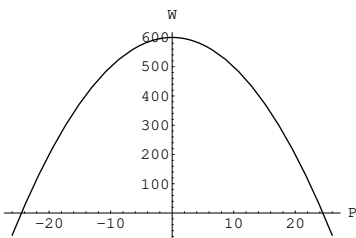


$$\begin{aligned}\text{slope} &= -\frac{5}{3} \\ \text{y-int} &= 4 \\ \text{x-int} &= \frac{12}{5}\end{aligned}$$

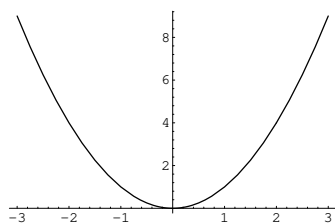
2. a.



2. b.

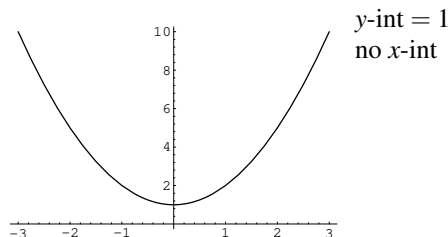


3. a.



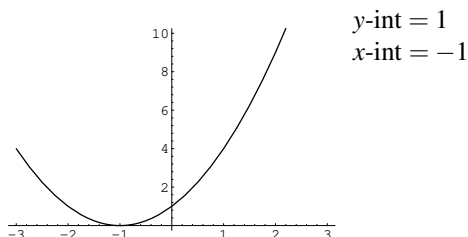
$$\begin{aligned}\text{y-int} &= 0 \\ \text{x-int} &= 0\end{aligned}$$

3. b.



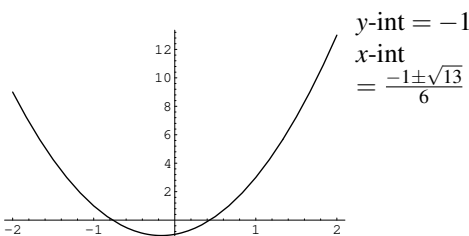
$$\begin{aligned}\text{y-int} &= 1 \\ \text{no x-int}\end{aligned}$$

3. c.



$$\begin{aligned}\text{y-int} &= 1 \\ \text{x-int} &= -1\end{aligned}$$

3. d.



$$\begin{aligned}\text{y-int} &= -1 \\ \text{x-int} &= \frac{-1 \pm \sqrt{13}}{6}\end{aligned}$$

4.

$$\begin{aligned}c(5, -3) &= 17 & j(17) &= 17 \\ j(c(3, -5)) &= 17 & s(1.1) &= 1.21 \\ Q(2) &\text{ is undefined} & Q(3/7) &= -\frac{13}{33} \\ H(1) &= 3 & H(7) &= 22 \\ r(s(-4)) &= \frac{1}{16} & r(Q(3)) &= \frac{3}{7} \\ c(a, b) &= 17 & j(u^2 + 1) &= u^2 + 1 \\ r(1/17) &= 17 & Q(0) &= -\frac{1}{6} \\ D(5, -3) &= 8 & D(-3, 5) &= -8 \\ H(4) &= 18 & H(H(H(-3))) &= 2 \\ Q(r(3)) &= -\frac{1}{3} & T(3, 7) &= 1\frac{1}{3}\end{aligned}$$

5. a. True. $r(r(x)) = \frac{1}{1/x} = x = j(x)$

5. b. True. If a number is greater than 1, then its square is greater than 1.

5. c. False. This can be shown by any number of “counterexamples”: $-1 > -3$, but $s(-1) = 1$ and this is less than $s(-3) = 9$.5. d. False. Again, there are many counterexamples: $s(3+4) = 49$, but $s(3) + s(4) = 25$ 5. e. False. $D(D(a, b), c) = (a - b) - c = a - b - c$, while $D(a, D(b, c)) = a - (b - c) = a - b + c$. There are many counterexamples to $a - b - c = a - b + c$: for example, $5 - 1 - 2 = 2 \neq 6 = 5 - 1 + 2$.

6. These numbers satisfy $\frac{2x+1}{3x-6} = \frac{3x-6}{2x+1}$ or

$$4x^2 + 4x + 1 = 9x^2 - 36x + 36.$$

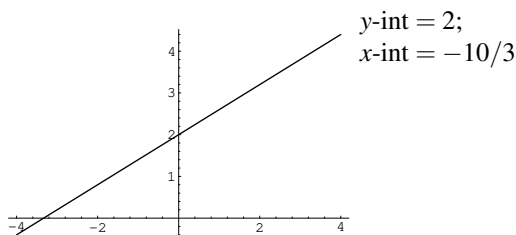
Solving this quadratic equation, we get $x = 1, 7$.

7. a. The natural domain of Q is all real numbers except 2; for H the natural domain is all real numbers.

7. b. Since the domain of r is all reals except 0, and the domain of Q is all reals except 2, the domain of P is all reals except 0 and $1/2$ (since $r(1/2) = 2$). The natural domain of R is all reals except 2 (where Q is undefined) and $-1/2$, since $Q(-1/2) = 0$ and r is not defined at 0.

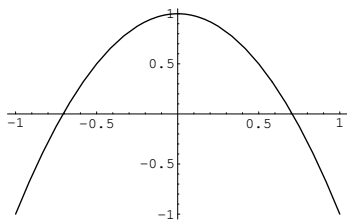
7. c. We must have $\frac{1-t^2}{t^2-4} \geq 0$. So the numerator and the denominator must have the same sign—or else the numerator must be 0. This happens only when $1 \leq t^2 < 4$. (If $t^2 < 1$ then the numerator is positive but the denominator is negative. If $t^2 > 4$ then the denominator is positive but the numerator is negative.) Finally, the condition $1 \leq t^2 < 4$ occurs when either $-2 < t \leq -1$ or $1 \leq t < 2$.

8. a.



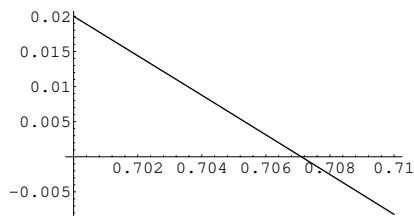
8. b. $y = f(-1) = 1.4$ and $y = f(2) = 3.2$. The differences are $\Delta y = 1.8$ and $\Delta x = 3$. Therefore, the slope is $\Delta y / \Delta x = 1.8/3 = .6$, and this is also what the formula says the slope ought to be.

9. a.

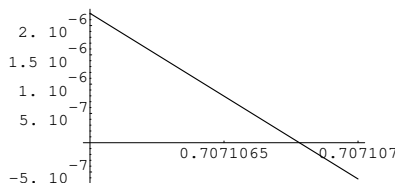


The y-intercept is 1, while the x-intercepts occur where $1 - 2x^2 = 0$, or $x = \pm 1/\sqrt{2}$.

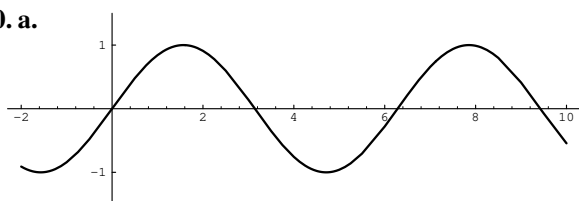
9. b.



We see that the graph crosses the x-axis about halfway between $x = .706$ and $.708$; thus $x = .707$ to 3 decimal places. The graph below shows a later stage in this process. We can estimate by eye that the x-intercept occurs at $x \approx .7071068$.



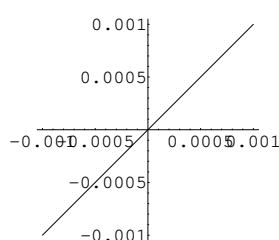
10. a.



There are four x-intercepts between -2 and 10 , at 0 and approximately 3.14, 6.28, and 9.42.

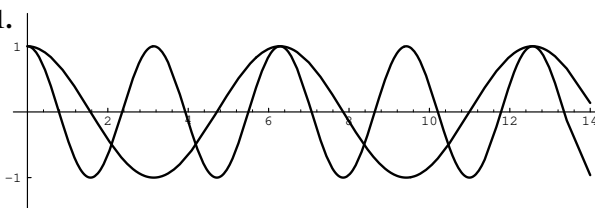
10. b. The largest value of $f(x)$ is approximately 1, and occurs when x is approximately 1.57 and 7.85. These are found by magnifying the graph appropriately.

10. c.



The graph appears to be a straight line whose slope is about 1.

11.

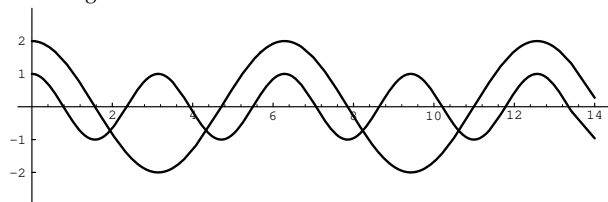


11. a. The space between the x-intercepts of $f(x) = \cos x$ is about $\Delta x = 3.14$. The space between the intercepts of $g(x) = \cos 2x$ is about half of that: $\Delta x = 1.57$.

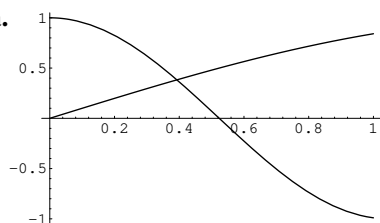
11. b. The pattern of $g(x)$ is about 3.14 units wide; the pattern of $f(x)$ is about 6.28 units wide.

11. c. The graph of $g(x)$ looks like a compressed version of the graph of $f(x)$ (or else $f(x)$ is a stretched version of $g(x)$).

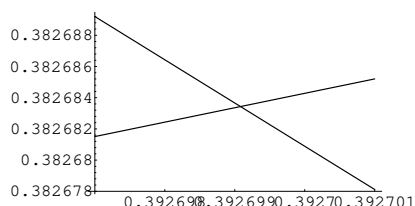
11. d. The function $h(x) = 2\cos x$ is sketched below with $g(x) = \cos 2x$. They have the same shape; h is twice the size of g .



12. a.

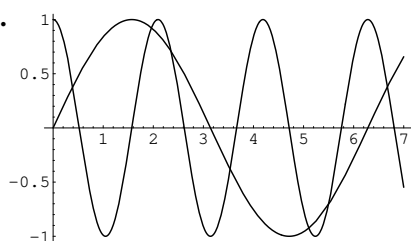


12. b.



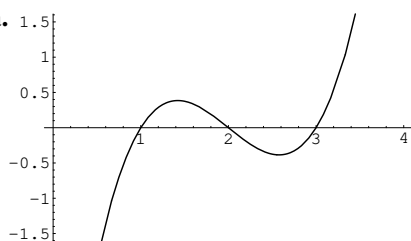
The figure above shows that the two graphs cross between $x = .392699$ and $x = .392700$.

12. c.



The two graphs cross frequently. The next crossing occurs where $x \approx 1.9634\dots$ since the graphs cross between $x = 1.9634$ and $x = 1.9635$. This value is obtained by successively magnifying the graphs near the new crossing point.

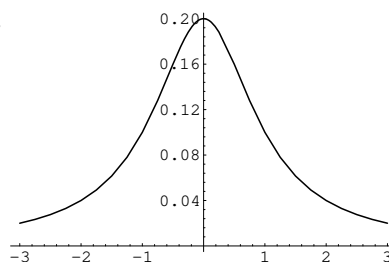
13. a.



Domain and range are both all real numbers. The graph crosses the w -axis at 1, 2, and 3, and the F -axis at -6 .

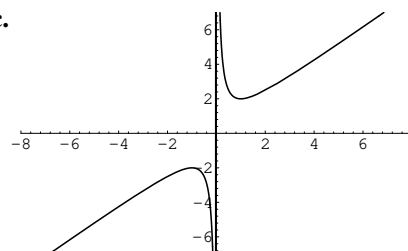
The graph has a local maximum near $w = 1.4$ and a local minimum near $w = 2.6$.

13. b.



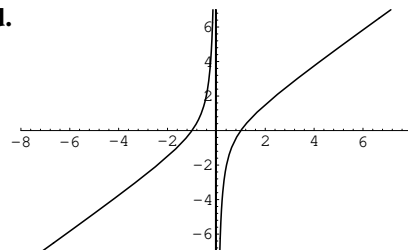
The domain is all real numbers, while the range is only those $0 < Q \leq 1/5$. The graph doesn't cross the a -axis (it lies always above it), but it crosses the Q -axis at $Q = 1/5$. There is a maximum at $a = 0$, but there is no minimum.

13. c.



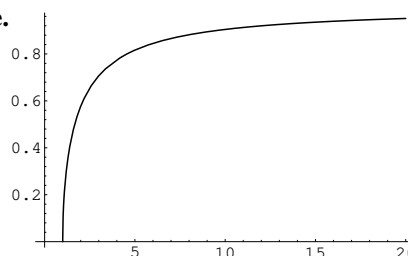
The domain is all $x \neq 0$, while that range is only those E with $2 \leq E$ or $E \leq -2$. The graph doesn't cross the x -axis or the E axis. There is a local maximum near $x = -1$ and a local minimum near $x = 1$.

13. d.



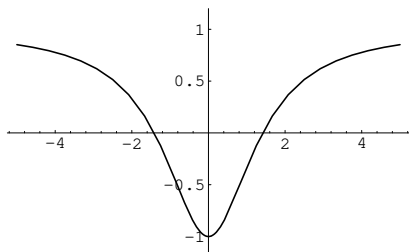
The domain is all $x \neq 0$, and the range is all real numbers. The graph crosses the x -axis at $x = \pm 1$ but it doesn't cross the e -axis. There is no minimum or maximum.

13. e.



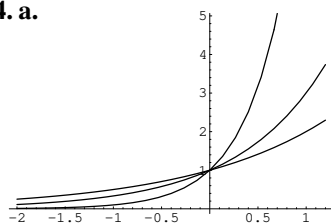
The domain is all $u \geq 1$, while the range is only those $0 \leq g < 1$. The graph crosses the u -axis at $u = 1$ but it doesn't cross the g -axis at all. There is a minimum at $u = 1$ but no maximum.

13. f.



The domain is all real numbers, while that range is only those M satisfying $-1 \leq M < 1$. The graph crosses the u -axis at $u = \pm 1/\sqrt{2}$ and it crosses the M -axis at $M = -1$. There is no maximum, but there is a minimum at $u = 0$.

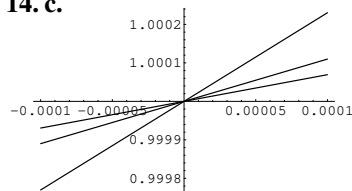
14. a.



The steepest graph is h and the gentlest is f . When $x = -2$, $f(x)$ has the largest value.

14. b. The graph that climbs most rapidly when $x = 0$ is $h(x)$.

14. c.



The graphs appear to be three straight lines, with slopes $h \approx 2.3$, $g \approx 1.1$, $f \approx .7$.

15. The answers here depend on the choice of x_0 . However, any answer must have the form indicated in each case below.

15. a. For any x_0 , $y_0 = -\frac{1}{2}x_0 + 3$. Moreover, $m = -\frac{1}{2}$, so $y - (-\frac{1}{2}x_0 + 3) = -\frac{1}{2}(x - x_0)$.

15. b. Here $m = \frac{2}{3}$ and $y_0 = (2x_0 - 7)/3$. Thus $y - (2x_0 - 7)/3 = \frac{2}{3}(x - x_0)$.

15. c. Here $m = -\frac{5}{3}$ and $y_0 = 4 - \frac{5}{3}x_0$. Thus $y - (4 - \frac{5}{3}x_0) = -\frac{5}{3}(x - x_0)$.

16. a. Take $x_0 = 2$, $y_0 = -5$. Then, to find $f(2.1)$, use $\Delta x = 2.1 - x_0 = .1$ and $\Delta y = m \cdot \Delta x = 3 \cdot .1 = .3$. Thus $y = f(2.1) = y_0 + \Delta y = -5 + .3 = -4.7$. Similarly,

$$f(2.0013) = -5 + 3 \times .0013 = -4.9961$$

$$f(1.87) = -5 + 3 \times -.13 = -5.39$$

$$f(922) = -5 + 3 \times 920 = 2755$$

16. b. Here we use a knowledge of Δy to determine Δx and then x . Take $x_0 = -1$ and $y_0 = 6$. If $G(x) = 8$, then

$\Delta y = 8 - y_0 = 2$ and so $\Delta x = \Delta y/m = 2/-2 = -1$. Hence $x = x_0 + \Delta x = -1 - 1 = -2$. In other words, $G(-2) = 8$. We tabulate this information below, and use it as a pattern to answer the remaining parts of the question.

$G(x)$	Δy	Δx	x
8	2	-1	-2
0	-6	3	2
5	-1	.5	-.5
491	485	242.5	241.5

17. $T = .05P$, where T is the sales tax, and P is the price of the object. The tax on the \$289.00 television set is \$14.45. The tax on the \$37.50 toaster is \$1.88.

18. W changes by -68 when Z changes from 3 to 7; by -6.8 when Z changes from 3 to 3.4, and by $-.34$ when Z changes from 3 to 3.02. $\Delta W = m\Delta Z$ for $m = -17$.

19. a. Since q is a linear function of p , there is some multiplier m for which $\Delta q = m\Delta p$. Using the first and fourth columns of the table, we find

$$\Delta q = -6 \quad \text{when} \quad \Delta p = 10.$$

Thus $m = \Delta q/\Delta p = -6/10 = -.6$.

p	-3	0	2	7	13	26/3	π
q	7	26/5	4	1	-13/5	0	$\frac{26}{5} - \frac{3}{5}\pi$

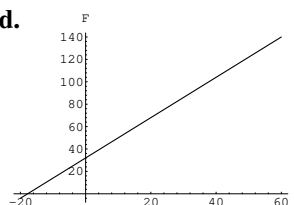
19. b. $\Delta q = -.6\Delta p$. We can find b in $q = -.6p + b$ by substituting known values for p and q . For example, $7 = -.6 \times (-3) + b$, so $b = 26/5 = 5.2$ and $q = -.6p + 5.2$.

20. a. A Celsius degree expresses a larger change in temperature.

20. b. It takes $\frac{9}{5}$ Fahrenheit degrees to make the temperature go up one Celsius degree. It takes $\frac{5}{9}$ of one Celsius degree to make it go up one Fahrenheit degree.

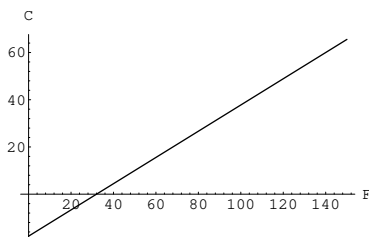
20. c. The multiplier m in the equation $\Delta F = m\Delta C$ is $\frac{9}{5}$. The multiplier μ in the equation $\Delta C = \mu\Delta F$ is $\frac{5}{9}$. The relation between μ and m is $m = 1/\mu$.

20. d.



$F = \frac{9}{5}C + 32$. The slope is $\frac{9}{5}$, and the vertical intercept is 32.

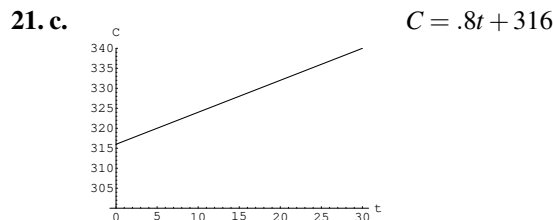
20. e. Here $C = \frac{5}{9}(F - 32)$. The graphs in parts (d) and (e) (see below) coincide if the graph in (e) is reflected in the line $C = F$ and the C and F scales are adjusted properly.



20. f. Yes. The temperature is the same if $\frac{9}{5}C + 32 = C$. This occurs when $C = -40^\circ$. This occurs in very cold climates, e.g., near the North and South poles, or on mountaintops in various places in the world.

21. a. Let $C(t)$ be the concentration of CO_2 t years after 1960. Then $\Delta C = m(\Delta t)$ where $m = .8$ PPM/year. If $5 = .8\Delta t$, then $\Delta t = 6.25$ years, and if $15 = .8\Delta t$, then $\Delta t = 18.75$ years.

21. b. In 1970, $\Delta t = 10$ years, so $\Delta C = .8$ PPM/year $\times 10$ years = 8 PPM. Since $C = C(0) = 316$ PPM in 1960, $C = C(10) = 324$ PPM in 1970 and $C = C(20) = 332$ PPM in 1980.



21. d. The answers for 1970 agree, but the concentration increase is 6 PPM more than expected for 1980.

21. e. Using $C(10) = 324$ and $C(20) = 338$ we have $m = (338 - 324 \text{ PPM}) / (10 \text{ years}) = 1.4$ PPM/yr, so C in 1990 $= C(30) = 352$ PPM and the increase from 1970 to 1990 would be estimated as 28 PPM.

21. f. Since $C = 316 + .8t$, $C = 0$ when $t = -\frac{316}{.8} = -395$ years. In other words, 395 years before 1960 (in 1565) there would have been no CO_2 in the atmosphere using this model.

22. a. When the temperature increases by 40° , the length increases by .0052 inches. A change in temperature of 20° (from 40° to 60°) is only $1/2$ as big as an increase of 40° . Since length changes in proportion to temperature, the length must increase by only .0052 inches $\times 1/2 = .0026$ inches. From 40° to 100° is an increase of 60° , or $3/2$ as much as the original increase. Thus the increase in length is $.0052 \times 3/2 = .0078$ inches.

22. b. The multiplier is $k = .00013$ inches/ $^\circ\text{F}$.

22. c. $\Delta L = .00013\Delta T$.

22. d. $\Delta T = -40$ so $\Delta L = -.0052$ and $L = 99.9948$ inches.

22. e. $L = 100 + .00013(T - 40) = .00013T + 99.9948$.

22. f. $\Delta L = .01 = .00013\Delta T$, so $\Delta T = \frac{.01}{.00013} \approx 76.9^\circ\text{F}$. Thus, a temperature of $T = 40 + 76.9 = 116.9^\circ\text{F}$ will make $L = 100.01$ in.

22. g. Since $k_{\text{aluminum}} = .00013$ in/ $^\circ\text{F}$, and $k_{\text{steel}} = .00067$ in/ $^\circ\text{F}$, the steel bar expands more than the aluminum bar when the temperature is increased.

22. h. When $T = 80^\circ\text{F}$, $L = 100.0268$ in.

23. The model suggests that $\Delta v = 32\Delta t$ ft/sec.

23. a. Using the equation $\Delta v = 32\Delta t$, and the fact that $\Delta t = 2$ when t changes from 1 to 3, we see $\Delta v = 32 \times 2 = 64$ ft/sec. Since $v = 40$ ft/sec when $t = 1$, $v = 40 + 64 = 104$ ft/sec now.

23. b. $\Delta v = 32\Delta t$.

23. c. We have $v = 32t + b$ for some b . Since $v = 40$ when $t = 1$ we see $40 = 32 + b$, so $b = 8$ and $v = 32t + 8$.

23. d. The rate d' at which the distance is increasing is the velocity. Thus $d'(1) = v(1) = 40$ ft/sec, $d'(2) = v(2) = 72$ ft/sec, and $d'(3) = v(3) = 104$ ft/sec.

23. e. d is not a linear function of t , since the rate of change of d is not constant.

24. In the equation $P' = kP$, k is the multiplier that relates P' linearly to P . Therefore, the units for k must be the units for P' divided by the units for P . The result follows because the units for P' are *persons per year* and the units for P are *persons*.

25. a. For Poland: $P' = .009P$; for Afghanistan: $A' = .0216A$.

25. b. The net growth rates are the values of P' and A' . These are

$$P' = .009 \times 37.5 = 337.5 \text{ thousand per year}$$

$$A' = .0216 \times 15 = 324 \text{ thousand per year}$$

Thus, Poland has the larger net growth rate. The assertion that the country with the larger per capita growth rate must have the larger net growth rate, is clearly false, as this example shows. The net growth rate depends on both the relative growth rate and the current population.

25. c. In Poland, there were 337,500 net births in 1 year. Thus, there must have been 1 birth every $1/337500$ of a year. Since there are 525,600 minutes in a year, we can also say there were 337,500 births in 525,600 minutes,

or 1 birth every 525600/337500 minutes, or 1 birth every 1.57 minutes (93.44 seconds).

In Afghanistan, there was 1 birth every 1/324000 of a year, or 1 every 525600/324000 minutes, or 1 every 1.62 minutes (97.3 seconds).

26. a. Use the relationship $P' = kP$ with $P = 32$ gm and $P' = .79$ gm/hr. Then $.79 = k \cdot 32$, so $k = .0247$ gm/hr per gm. Thus, $P' = .0247P$ gm/hr.

26. b. Since $P' = .79$ gm/hr and $\Delta t = 1/60$ hr, we find $\Delta P = P' \Delta t = .79 \times 1/60 = .0132$ gm.

If we want $\Delta P = .5 = .79 \Delta t$, then $\Delta t = .5/.79 = .63$ hr, or about 38 minutes.

27. $R' = -\frac{1}{2337}R$. The negative sign for the constant of proportionality indicates that decay, rather than growth, is taking place.

28. a. Use $C' = k(C - 70)$. Since the coffee is cooling we can assume $C \geq 70$, so $C - 70 > 0$. But we need $C' < 0$ since C is decreasing, so we must choose $k < 0$.

28. b. If $C = 180^\circ\text{F}$ and $C' = -9^\circ\text{F/min}$, then $-9 = k(180 - 70) = k \cdot 110$. Hence $k \approx -.082^\circ\text{F/min per }^\circ\text{F}$.

28. c. If $C = 120^\circ\text{F}$, $C' = (-.082)(120 - 70) = -4.1^\circ\text{F/min}$.

28. d. Using $C' = -9^\circ\text{F/min}$ (obtained from our calculations when $C = 180^\circ\text{F}$), we would have $\Delta C \approx C' \Delta t$, or $\Delta t = -60/-9 = 6.67$ min.

Using $C' = -4.1^\circ\text{F/min}$ (obtained from our calculations when $C = 120^\circ\text{F}$), we would have $\Delta C \approx C' \Delta t$, or $\Delta t = -60/-4.1 = 14.6$ min.

A better estimate would be the average of these two times, or about 10.65 minutes.

1.3 Using a Program

1. The program should run and produce four lines of output.

2. a. Lines 2–4.

2. b. Line 5.

2. c. Lines 11–13.

2. d. Lines 14–17.

2. e. Line 18.

2. f. Lines 8–18. (Do not include lines 7 and 19 in this list!)

2. g. Line 7.

3. The program runs and prints one line of output containing the input values of t , S , I , and R .

4. The program runs and prints *two* lines of output: values of t , S , I , and R when $t = 0$ and then when $t = 1$.

5. The program runs and prints two lines of output, as in the previous exercise. However, both lines contain the values of t , S , I , and R when $t = 0$. This happens because the second PRINT command is given *before* the values are updated.

6. This program runs, and it does exactly the same thing as the original program! The only change is in the counter k , the computations themselves are unaffected. As k goes from 26 to 28, the calculations for days 1, 2, and 3 are carried out.

7. Program 1 prints

3 3,
meaning that $A = 3$ and $B = 3$.

8. Program 2 replaces the input value of B with the input value of A (which is 2), and prints

2 2,
meaning that $A = 2$ and $B = 2$.

Program line	Value of variable	
	A	B
$A = 2$	2	undefined
$B = 3$	2	3
$A = A + B$	5	3
$B = A + B$	5	8
The program prints	5	8.

10. Program 4 prints out

1 8 27 64 125

but in a column rather than a row, as here.

11. In Program 5, the PRINT statement has been moved outside the FOR-NEXT loop. Thus the value of k^3 will be printed only once, using the loop exit value of $k = 5$. The output is 125.

12. 13. 14. The outputs are as follows:

	Program 6	Program 7	Program 8
1	1	1	1
2	3	2	3
3	6	3	6
4	10	4	10
5	15	5	15

15. In Program 8, the PRINT statement is inside the loop; in Program 9, it is outside (so only the final calculated values of A and B will be printed by Program 9). The output of Program 9 is: 55 89

16. Change the FOR statement to:

FOR k = 1 TO 6.

Then, to 1 decimal place accuracy,

t	S	I	R
0	45400.0	2100.0	2500.0
1	44446.6	2903.4	2650.0
2	43156.1	3986.5	2857.4
3	41435.7	5422.1	3142.1
4	39189.0	7281.5	3529.4
5	36335.5	9615.0	4049.5
6	32841.8	12421.9	4736.3

17. a. $S = 133.1, I = 10329.7, R = 39537.2$.

17. b. The largest value of I (28578.63) appears on day 13.

18. After line 10, insert:

PRINT t, Sprime, Iprime, Rprime

This will print the correct values.

19. Use the output from the last exercise to determine the value of $-S'$, which is the rate at which *new* infections appear. The largest value of $-S'$ is 4609.1, which occurs when $t = 8$.

20. Inspect the values of R' and find that R' takes its largest value (2041.3) when $t = 13$. There is a direct connection with question 17b: since $R' = \frac{1}{14}I$, R' will have its largest value when I has its largest value.

21. I grows most rapidly when $t = 7$ (and $I' = 3375.7$); it declines most rapidly when $t = 18$ (and $I' = -1398.4$).

22. a. Change line 5 to `deltat = -1`. Then

t	S	I	R
0	45400.0	2100.0	2500.0
-1	46353.4	1296.6	2350.0
-2	46954.4	788.2	2257.4
-3	47324.5	474.4	2201.1

The values of S , I , and R here, when $t = -1$, agree with those in the text.

22. b. The previous table gives the values of S , I , and R when $T = -3$.

23. Use the program as modified in the previous question, and go through the loop at least 15 times. When $t = -15, I = .9$, so the infection can be said to have started 15 days ago.

24. Modify SIR to take the values of S , I , and R at time $t = 2$ as *input* values, and use a negative value

for `deltat`. Do this for `deltat = -1` and for `deltat = -2`.

25. *Ad libitum*; the answer to this question depends on free choices made by the student.

1.4 Chapter Exercises

1. Since the line contains the points $(N, Y) = (40, 750)$ and $(180, 0)$, its slope is $\Delta Y / \Delta N = -750/140 = -75/14$ and its equation is

$$Y(N) = -\frac{75}{14}(N - 180)$$

on the interval $40 \leq N \leq 180$. Thus

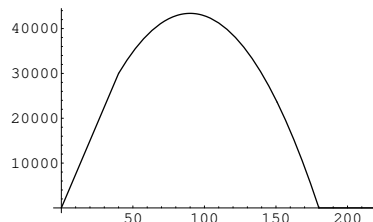
$$T(N) = Y(N) \cdot N = -\frac{75}{14}(N^2 - 180N)$$

on the same interval.

2. The complete expression for $T(N)$ is given by

$$T(N) = \begin{cases} 750N & 0 \leq N \leq 40 \\ -\frac{75}{14}(N^2 - 180N) & 40 < N < 180 \\ 0 & 180 \leq N \end{cases}$$

The graph is a straight line with positive slope for $0 \leq N \leq 40$. For $40 < N < 180$, the graph is part of a downward-opening parabola with its peak at $N = 90$. The graph hits the N -axis at $N = 180$ and thereafter lies along the axis for $N > 180$.



3. The maximum total yield is $T(90) = 43392\frac{6}{7}$ (or $\frac{75 \times 90^2}{14}$) pounds of apples, at 90 trees per acre.

4. The points $(P, 750)$ and $(Q, 0)$ are on the graph of Y , so

$$Y(N) = \frac{750}{P-Q}(N-Q) \quad \text{when } 0 \leq N \leq P$$

Therefore

$$T(N) = \begin{cases} 750N & 0 \leq N \leq P \\ \frac{750}{P-Q}N(N-Q) & P \leq N \leq Q \\ 0 & Q \leq N \end{cases}$$

The maximum value of T is $T(Q/2) = 9(\frac{750}{P-Q})(Q/2)^2$ provided $P \leq Q/2$. However, if $P > Q/2$, the maximum value of T is $T(P) = 750P$.

Therefore, the best strategy is to plant either $Q/2$ or P trees per acre—whichever is the larger number.

5. We construct a table of values of R and R' , using $R' = -\frac{1}{2337}R$ gm/year. Since $\Delta t = 10$, $\Delta R = 10R'$. We get

t	R	R'
0	.0720	-.00003
10	.0717	-.00003
20	.0714	-.00003
30	.0711	-.00003
40	.0708	

and this implies $R = .0708$ gm after 40 years.

6. We construct a table of values of P , A , P' , and A' , using $\Delta t = 1$.

t	P	P'	A	A'
1985	37 500 000	337 500	15 000 000	324 000
1986	37 837 500	340 538	15 324 000	330 998
1987	38 178 038	343 602	15 654 998	338 148
1988	38 521 640	346 695	15 993 146	345 452
1989	38 868 335	349 815	16 338 598	352 914
1990	39 218 150		16 691 512	

7.

t	d	v	d'	v'
0	0	10	10	32
1	10	42	42	32
2	52	74	74	32
3	126	106		

Solutions: Chapter 2

Successive Approximations

2.1 Making Approximations

There and Back Again

1. a. The *exact* value of $I(1)$ is 3062.1...

1. b. The exact value of $R(1)$ is 2682.3...

2. a. & b. The discrepancy and Δt appear to be roughly proportional. Since $\Delta t = .0001$ produces a discrepancy of about .0336, we can say $\text{discrepancy} \approx 336\Delta t$. Thus, to make the discrepancy .001, choose $\Delta t = .001/336 \approx .000003$.

3. a. Do this in two stages. First, get SIRVALUE to step three days into the future, given $\Delta t = .01$, by setting

```
tfinal = 3
numberofsteps = 300
```

Second, get SIRVALUE to go backward by replacing the initial values of S , I , and R with the day 3 values (just obtained) and by setting

```
tinitial = 3
tfinal = 0
numberofsteps = 300
```

3. b.	Δt	$S(3)$	$I(3)$	$R(3)$
	1.0	41435.7	5422.14	3142.13
	0.1	40578.4	6121.25	3300.33
	0.01	40472.2	6207.29	3320.52
	0.001	40461.3	6216.09	3322.59

Δt	new $S(0)$	discrepancy
1.0	46261.5026	861.5026
0.1	45493.4026	93.4026
0.01	45409.4605	9.4605
0.001	45400.9292	0.9292

3. c. The discrepancy in this table is roughly $930\Delta t$. In the original table on page 66 of the text, the discrepancy is roughly $336\Delta t$, so the discrepancy here is about 3 times as large as it was for the table in the text.

3. d. When $\Delta t = .001$ the hundreds place of $S(3)$ is stable, so a conservative choice for the stabilization of the integer part of $S(3)$ would be $\Delta t = .00001$. When $\Delta t = .001$ the tens place of $R(3)$ is stable, so $\Delta t = .0001$ would be an appropriate estimate for the stabilization of the integer part of $R(3)$.

4. Use the formula on page 72 of the text: $S_3(1.7) = 43543.25$ and $S_3(2.5) = 42295.9$.

5. To find $S_{3000}(3)$ let `numberofsteps = 3000`, then $S_{3000}(3) = 40461.3$, $I_{3000}(3) = 6216.09$ and $R_{3000}(3) = 3322.59$.

6. Running the loop until $k = 192$ gives the value of S at the *end* of the 192nd segment, which is the value of S at the *beginning* of the 193rd segment. However the value of the slope of the 193rd segment is calculated during the 193rd trip through the loop, after S, I, R are updated. Therefore one way to get what we want is this: change `FOR k = 1 to numberofsteps` to `FOR k = 1 to 193` and place a `PRINT` statement that reads `PRINT t, S, Sprime` after the definition of `Sprime`, which would instruct the computer to print out 193 lines of information. A second way is to calculate by hand yourself the value of S' of the 193rd segment, using the value of S printed out after the 192nd loop, in which case change `FOR k = 1 to numberofsteps` to `FOR k = 1 to 192` and leave the `PRINT t, S, I, R` statement outside the loop.

Using `tfinal = 3`, `numberofsteps = 300` and running the loop `FOR k = 1 TO 192`, $S_{300}(1.92) = 42803.26$, and therefore $S' = -1830.32$.

7. a. To find the value of $S_{300}(2.84135)$, we would need to look at the 285th segment, since $\Delta t = .01$ and $2.84 < 2.84135 < 2.85$.

7. b. At the beginning of this segment, $t = 2.84$, $S = 40864.94$ and $S' = -2404.29$.

7.c. The equation of the 285th segment is $S(t) = 40864.94 - 2404.29(t - 2.84)$.

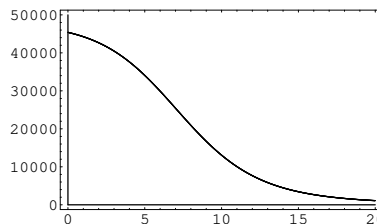
7.d. $S_{300}(2.84135) = 40861.694$.

8. To modify SIRVALUE to calculate estimates for S , I , and R when $\Delta t = .05$ and $t = -6$, set `numberofsteps = 120` since $\frac{6}{.05} = 120$ and `tfinal = -6`. This gives estimates of $S(-6) = 47658.5$, $I(-6) = 188.411$, and $R(-6) = 2153.13$.

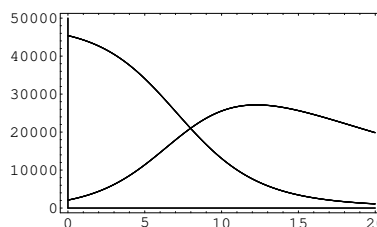
9.a. To use SIRPLOT to look at the graph of $S(t)$ over the first 20 days with $\Delta t = .01$, set `tfinal = 20` and `numberofsteps = 2000`. If the WINDOW was set in your original program, you must now reset it to see $0 \leq t \leq 20$ and $0 \leq S \leq 50000$.

9.b. Here is Mathematica code that produces the graph of $S(t)$ shown immediately after the code.

```
tinit = 0;
tfin = 20;
t = tinit;
S = 45400;
Inf = 2100;
R = 2500;
NUM = 2000;
deltat = (tfin - tinit)/NUM;
Curve = {};
For[k = 1, k <= NUM, k++,
  Sp = -.00001 S * Inf;
  Ip = .00001 S * Inf - Inf/14;
  Rp = Inf/14;
  deltaS = Sp * deltat;
  deltaI = Ip * deltat;
  deltaR = Rp * deltat;
  Curve = Append[
    Curve, Line[{
      {t, Inf},
      {t + deltat, Inf + deltaI}
    }
  ];
  t = t + deltat;
  S = S + deltaS;
  Inf = Inf + deltaI;
  R = R + deltaR
];
Show[
  Graphics[{
    Curve, Line[{{0,0},{20,0}}],
    Line[{{0,0},{0,50000}}]
  ]],
  PlotRange -> All, Frame->True
]
```



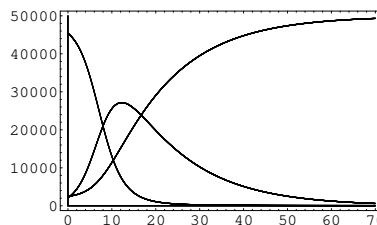
9.c. The following shows the graph of $I(t)$ with the graph of $S(t)$; it is obtained by adding code that plots the line from (t, I) to $(t + \text{deltat}, I + \text{deltaI})$. Note that $I(t)$ peaks on roughly the 13th day.



9.d. To use SIRPLOT to plot the graphs over the first 70 days with $\Delta t = .01$, make the following changes:

```
tfinal = 70
numberofsteps = 7000
```

Also reset the window (if it was set in the original program) to view $0 \leq t \leq 70$, and add appropriate code to plot the line from (t, R) to $(t + \text{deltat}, R + \text{deltaR})$.



The DO-WHILE loop

10. Using DO-WHILE as modified, you get $t = 12.27$.

11.a. The peak value is $I(12.27) = 27160.50$. (Note that $I(12.28) = 27160.47$.) Get $I(12.27)$ from SIRVALUE using `tfinal = 12.27` and `numberofsteps = 1227`.

11.b. $S(12.27) = 7129.32$. The value predicted in chapter 1 was 7143; it is slightly different. [Note: $S(12.28) = 7109.95$.]

12. If $S(0) = 5400$, then $I' < 0$, so the loop halts immediately. It prints the value of t as 0.

13. a. In order to determine how long the infection lasts, using the condition $I \geq 1$, replace `FOR k = 1 TO numberofsteps` with `DO WHILE I >= 1` and replace `NEXT k` with `LOOP` and let `deltat = .1`.

13. b. $t = 160.1$.

13. c. If `deltat = .01` then $t = 160.57$. The actual duration would probably be a bit more than 160 days, given the trend.

14. a.

Δt	days before $I(t) \leq 1$
-1.0	14.0
-0.1	18.5
-0.01	18.9

The infection arrived approximately 19 days ago.

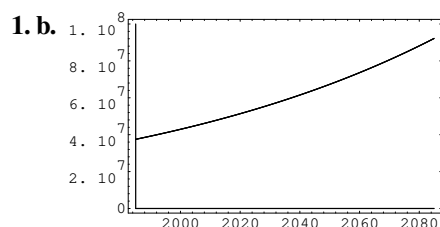
14. b. There were approximately 2120 recovered people at the start of the epidemic.

2.2 Mathematical Implications—Euler's Method

Approximate Solutions

1. a. We want to find $P(100)$. The following table shows $P(100) \approx 92.23$ million people.

Δt	$P(100)$
.1	92,197,800
.01	92,231,400
.001	92,234,700



Note: the above graph was created using a modification of the earlier Mathematica program with $\Delta t = .01$.

The Logistic Equation

2. According to the following table, $y(37) = 817.988\dots$, so $y(37) = 817.99$ to 2 decimal places accuracy.

Δt	$y(37)$
.1	817.61110
.01	817.95057
.001	817.98432
.0001	817.98769
.00001	817.98803
.000005	817.98805

3. a. y approaches 1000 as t gets large.

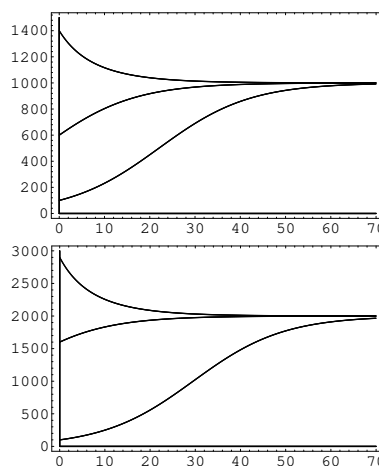
3. b. If $y(0) = 1000$, then $y' = 0$ (by calculation), so $\Delta y = 0$, so y will remain at 1000. The graph of $y(t)$ will be the horizontal line $y = 1000$.

3. c. If $y(0) = 1500$, the population will decrease over time, stabilizing at ≈ 1000 , since y' is negative for $y > 1000$.

3. d. If $y(0) = 0$, nothing will happen. If there are no rabbits put into the field, there will never be any rabbits in the field. If $y = 0$ then $y' = 0$, that is, there is no spontaneous generation of rabbits.

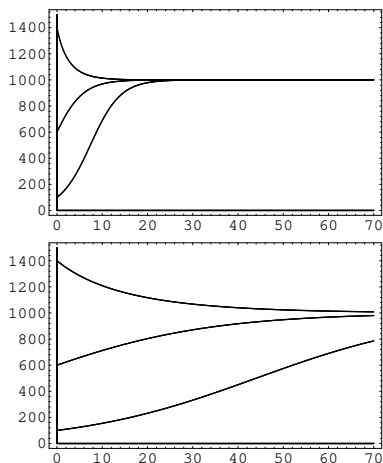
3. e. The *carrying capacity* represents the maximum number of rabbits that the field can support in the long run.

4. As the carrying capacity is increased the curve continues to increase past the old carrying capacity and levels off at the new capacity—at least if $y(0)$ is less than the carrying capacity. Likewise if $y(0)$ is greater than the carrying capacity then the population will decrease and stabilize at the new carrying capacity.



Note: the above graphs were created using a modification of the earlier Mathematica program with $\Delta t = .1$. The first set of solutions has a carrying capacity of 1000; the second, 2000.

5. As the constant $c = .1$ increases, the solution $y(t)$ approaches the carrying capacity at a faster rate, and as c decreases $y(t)$ approaches the carrying capacity at a slower rate, that is, the time it takes to reach the carrying capacity increases.



Note: the above graphs were created with $\Delta t = .1$. For the first set of solutions $c = .3$; the second, $c = .05$.

6. Using a DO WHILE $y < 900$ loop, with $\text{deltat} = .001$, $y(43.945) = 900.005$.

7. Using trial and error on the constant in the program SIRVALUE, you should find the constant to be about .2, or better yet, about .219. Alternatively, use DO-WHILE with the condition DO WHILE $y < 900$.

Using SEQUENCE

8. `numberofsteps = 2 ^ (j - 1)` controls the number of segments in an approximation. The fifth segment has 2^{5-1} or 16 segments. The fourteenth has 2^{14-1} or 8192 segments.

9. The slope of the first graph is $y' = 9$. This is obtained from the rate equation

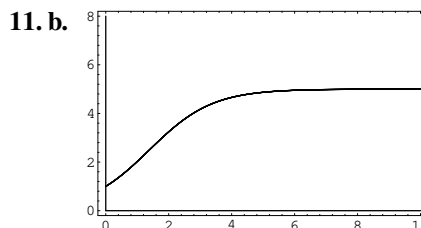
$$y' = .1y(1 - y/1000)$$

using the initial condition $y = 100$. The slope of the first segment of the second graph is *also* $y' = 9$. The slope of the second segment is $y' = 24.609375$.

10. In the original program SEQUENCE, the number of segments in the graph doubles with each successive approximation: 1, 2, 4, 8, ..., 1024, 2048, 4096, 8192 = 2^{14-1} . Furthermore, the length of an individual segment is cut in half from one stage to the next. In the modified program (where `numberofsteps = j`), the last (i.e.,

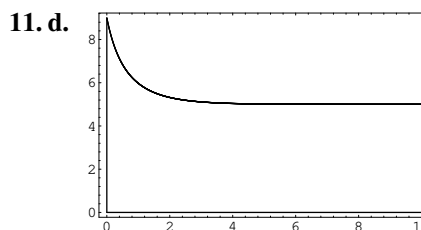
the fourteenth) graph contains only 14 segments, and so is only as good an approximation as the fifth graph in the original program. But the original program demonstrates that about 1000 segments are needed in order to get a good approximation; this explains why the modified program gives such poor results.

11. a. $y(10) = 4.999 \dots$

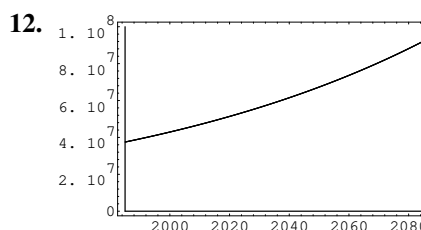


The right half of the graph levels off, becoming a horizontal line with slope 0. [Note: This is the 13th graph in the sequence of graphs.]

11. c. $y(50) \approx 5$. By the time $t = 50$, $y(t)$ is essentially a horizontal line and we can see that $y' = 0$ if $y = 0$ or $y = 5$.



Here $y(10) = 5.000 \dots$ [Note: This is the 13th graph in the sequence of graphs.]



Here $P(0) = 37.5$ million people and $P(100) \approx 92.2347$ million people.

13. $y(2) = 3.999 \dots$, accurate to 3 decimal places. By the 14th approximation, the third decimal place has stabilized to a 9. Specifically, when

`numberofsteps = 2 ^ 12`, $y(2) = 3.999023$;

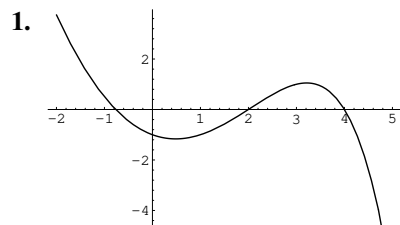
`numberofsteps = 2 ^ 13`, $y(2) = 3.999369$.

14. $y(1) = 3.141 \dots$, accurate to 3 decimal places. By the 14th approximation the third decimal place has stabilized to a 1. Specifically, when

`numberofsteps = 2 ^ 12`, $y(1) = 3.141837$;

`numberofsteps = 2 ^ 13`, $y(1) = 3.141715$.

2.3 Approximate Solutions



There are three roots altogether. From the graph it appears that 2 and 4 are roots. This can be checked exactly: for example, $4^2 - 2^4 = 16 - 16 = 0$, showing 4 is a root. By magnification, it can be seen that $-0.7666\dots$ is the third root.

2. The program has 3 lines of output. The first two lines are the lengths of the segments; the third, the sum of the lengths.

3. `DEF fnf (x) = x ^ 2` gives the instruction to work with the function $f(x) = x^2$, while `n = 2` indicates there are 2 segments to be measured.

4. The left-endpoints of the segment are designated by:

```
x1 = xinitial + (k - 1) * deltax
y1 = f(x1).
```

The right endpoints are designated by:

```
xr = xinitial + k * deltax
yr = f(xr).
```

5. The length of the k -th segment is calculated in the line:

```
segment = SQR((xr - x1) ^ 2 + (yr - y1) ^ 2)
```

The segment is the hypotenuse of the right triangle with base $(xr - x1)$ and altitude $(yr - y1)$.

6. To run the program using 20 segments, change `n = 2` to `n = 20`. The resulting estimated value is 1.478756512.

7.	Number of line segments	Sum of their lengths
	2	1.460 404 813 240 95
	20	1.478 756 512 029 07
	200	1.478 940 994 153 96
	2 000	1.478 942 838 910 70
	20 000	1.478 942 857 358 26
	200 000	1.478 942 857 542 75
	2 000 000	1.478 942 857 544 60

8. The length of the parabola $y = x^2$ on the interval $0 \leq x \leq 1$ is 1.47894286 to 8 decimal places, and 1.478942857545 to 12 decimal places. It appears that the LENGTH algorithm stabilizes at least two digits each

time as we increase the number of segments by a factor of 10. Since we have 9 places to the right of the decimal stable using 20,000 steps, we can conclude that using 2,000,000 steps gives us about 13 stable decimal places. Rounding this answer to 12 decimal places gives the second estimate, above.

9. Use trial and error with `numberofsteps` fixed at 20000, you should find $x \approx 3.041$.

x -coordinate	Sum of the lengths
3.04	9.991977
3.042	10.004305
3.0414	10.000606

10. The length of the curve $y = x^3$ on $0 \leq x \leq 1$, correct to 8 decimal places, is 1.54786565....

Segments	Total length
10	1.546 816 956 6
100	1.547 855 156 8
1000	1.547 865 549 7
10000	1.547 865 653 7
100000	1.547 865 654 7

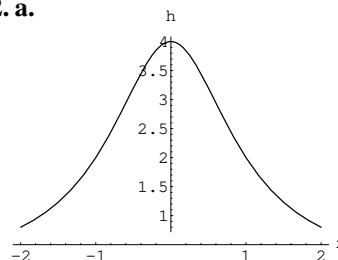
11. a. The length of the curve is 3.14076..., or approximately 3.1408.

11. b. Without rounding the second decimal place, the length stabilizes to 3.14... at 65 segments.

Segments	Total length
64	3.139967
65	3.140004
66	3.140040

11. c. The answer to eight places of accuracy is 3.14159265....

12. a.



Note: the x -axis has been shifted up to the line $y = .5$.

12. b.	Segments	Total length
	100	7.715 765 549 6
	1000	7.716 737 326 0
	10000	7.716 747 046 5
	100000	7.716 747 143 6
	1000000	7.716 747 144 6

The length of the curve on $-2 \leq x \leq 2$ to 8 decimal places of accuracy is 7.71674714....

13. Segments	Total length
100	3.815 282 726 1
1000	3.820 148 518 7
10000	3.820 197 296 3
100000	3.820 197 784 1
1000000	3.820 197 789 0

The length of the curve accurate to 8 decimal places is 3.82019778....

The program BABYLON

14. The output of Babylon should be as given in the text.

15. The estimate 2.23606797749979 is obtained at the fourth step. Its square is 5.000000000000001.

16. a. Using 14-digit precision the estimate for $\sqrt{80}$ stabilizes at the eighth step.

16. b. Using 14-digit precision, if $x = 9$, then the estimate for $\sqrt{80}$ stabilizes at the third step at 8.944271909999159. The final values in (a) and (b) are the same.

17. Using 2 as the initial estimate, the estimate for $\sqrt{250}$ stabilizes at the eighth step at 15.81138830084190. The estimate for $\sqrt{1990}$ stabilizes at the ninth step at 44.60941604639093. Using the integer nearest to the final answer both estimates stabilizes at the third step.

$$15.81138830084190^2 = 250.0000000000001$$

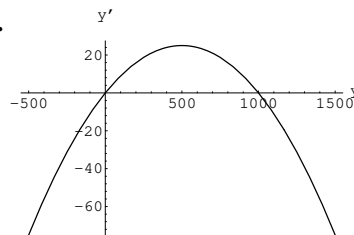
$$44.60941604639093^2 = 1990.000000000001.$$

18. Yes, for instance when $x = 2$ the estimates for $\sqrt{80}$ are given by the following table. It demonstrates that the Babylonian algorithm is much faster than SIRVALUE in converging to a limit.

n	$\sqrt{80}$
1	21
2	12.40476...
3	9.42694...
4	8.95662...
5	8.944280434131398
6	8.944271910003220
7	8.944271909999159

2.4 Chapter Exercises

1. a.



The graph is an ordinary parabola.

1. b. The graph of y' versus y takes on its largest value when $y = 500$. Thus, $y(t)$ will be increasing most rapidly as a function of t when $y = 500$. Therefore the graph of $y(t)$ versus t will be steepest when $y = 500$.

1. c. Let y_0 denote a y -value at which the graph of y' versus y crosses the y -axis. Then $y' = 0$ when $y = y_0$. There are two such values: $y_0 = 0$ and $y_0 = 1000$. If we consider y versus t and take $y = y_0$ as the initial value, then the graph of y versus t will be the horizontal line $y(t) = y_0$. In other words, the value of y will not change from its initial value.

2. The modified loops are:

```
FOR k = 0 TO 15
  Plot the line from (-20 + 10 * k, 250)
    to (-20 + 10 * k, 750)
NEXT k
FOR k = 0 TO 20
  Plot the line from (-20, 250 + 25 * k)
    to (120, 250 + 25 * k)
NEXT k
```

3. To plot the graphs over the first 120 days with $\Delta t = .1$, let $t_{\text{final}} = 120$, $\text{numberofsteps} = 1200$ and add the code to plot the additional lines for $I(t)$ and $R(t)$.

The code to produce the grid is as follows:

```
FOR k = 0 TO 25
  Plot the line from (5*k, 0) to (5*k, 50000)
NEXT k
FOR k = 0 TO 26
  Plot the line from (0, 2000*k) to (120, 2000*k)
NEXT k
```

Solutions: Chapter 3

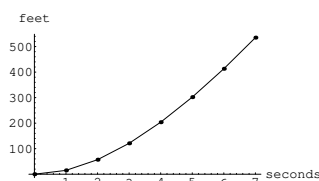
The Derivative

3.1 Rates of Change

A Falling Object

1. The velocity in this problem is measured in feet per second.

2. a.



2. b. The slope of any line in the graph has the units $\Delta D/\Delta t = \text{feet/second}$. These are units that measure velocity.

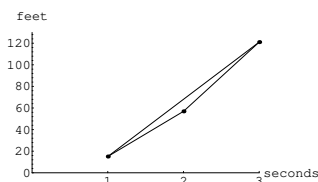
3. Here are the three calculations:

i) $\Delta D/\Delta t = (56.90 - 15.07)/(2 - 1) = 41.83 \text{ feet/second}$ from 1 to 2 seconds.

ii) $\Delta D/\Delta t = (121.03 - 56.90)/(3 - 2) = 64.13 \text{ feet/second}$ from 2 to 3 seconds.

iii) $\Delta D/\Delta t = (121.03 - 15.07)/(3 - 1) = 52.98 \text{ feet/second}$ from 1 to 3 seconds.

4. a.



Segment (ii) is the steepest, and segment (iii) has a slope in between those of (i) and (ii).

4. b. The third is probably the best estimate, since it lies between the 1–2 second estimate, which is undoubtedly too small, and the 2–3 second estimate which is clearly too large. Also, this 1–3 second estimate incorporates information on both sides of $t = 2$ seconds.

5. To estimate the velocity at $t = 4$ seconds, use the positions at 3 and 5 seconds:

$$\text{velocity} = \frac{302.0 - 121.03}{5 - 3} = 90.5 \text{ feet/sec.}$$

6. The object is speeding up as it falls, since its velocity is growing larger. For instance, we estimate its velocity at $t = 1$ sec to be about 53 feet/sec, and its velocity at $t = 4$ sec to be about 90.5 feet/sec.

7. Use data for 6 and 7 seconds to estimate the velocity at 7 seconds. Thus, velocity is

$$(535.10 - 413.16)/(7 - 6) = 121.94 \text{ feet/sec.}$$

So, between 7 and 8 seconds, we would expect the object to fall about 121.94 feet. Thus after 8 seconds, the object should have fallen about $535.10 + 121.94 = 657.04$ feet. This estimate is too low, because we are underestimating the velocity in the last second.

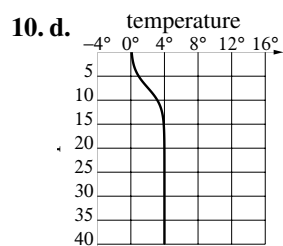
8. *Ad libitum*; answers here vary from institution to institution. However, units for inflation rate are dollars per year per hundred dollars—or just percent change per year (part b). Units for *change* in inflation rate are therefore percent change per year per year (part c).

9. *Ad libitum*; answers depend on free choices made.

10. a. Units are degrees Celsius per meter.

10. b. The rate is negative, because the temperature decreases as the depth increases. This is true in spite of the fact that the slope of the graph *appears* to be positive. However, the input axis is vertical and the output, horizontal. If you reposition the graph in the usual orientation, its slope is indeed seen to be negative—in agreement with the fact that the rate is negative.

10. c. Some variation in answers is to be expected here. At a depth of 0 meters, the temperature is changing about -1°C in 250 meters, or $-.004^\circ\text{C/m}$. At 500 meters, the temperature is changing about -4°C in 200 meters, or $-.02^\circ\text{C/m}$. Around 1000 meters, the temperature stabilizes at 4°C , so the rate of change is 0°C/m .



This graph measures depth from the ice-water interface. At that interface the temperature is 0°C. Below that it is reasonable to think the temperature rises rather rapidly to the stable level of 4°C.

10. e. When the graph is oriented as shown, the depth axis points directly down—in the direction in which depth actually increases.

3.2 Microscopes and Local Linearity

Using a Microscope

1. a. Estimate $f'(1)$ as

$$\frac{(f(1 + \Delta x/2) - f(1 - \Delta x/2))}{\Delta x}.$$

With $\Delta x = .002$, $f'(1) \approx -3.999948$. Subsequent parts of this question are done the same way.

1. b. $g'(0) \approx 2.302825$.

1. c. The slope at $t = 7$ is about .9945965.

1. d. The slope at $z = 0$ is 1.0000.

2. These are all obtained by constructing a sequence of slope values of the form

$$\text{slope} = \frac{f(a + \Delta x) - f(a - \Delta x)}{2\Delta x},$$

for the function $f(x)$ at $x = a$. Students should produce tables similar to the following.

2. a.	Δx	slope	With two decimal places stabilized, $f'(1) = -4.0$.
	1/2	-3	
	1/4	-3.75	
	1/8	-3.9375	
	1/16	-3.984375	
	1/32	-3.99609375	
	1/64	-3.9990234375	

2. b. $h'(0) = 1.1$.

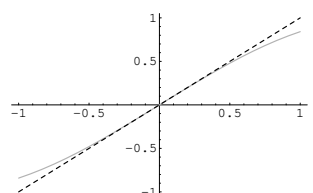
Δs	slope
1/2	1.1547
1/4	1.11248
1/8	1.10207
1/16	1.09948
1/32	1.09883
1/64	1.09866

2. c.	Δz	slope	$w' = .7$
	.1	.59501	
	.01	.70593	
	.001	.70709	
	.0001	.70711	
	.00001	.70711	

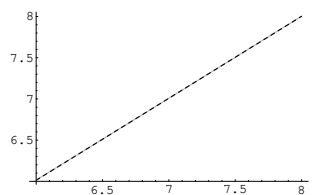
2. d.	Δt	slope	$y' = 1.0$
	1/2	.994476	
	1/4	.9945576	
	1/8	.9945780	

2. e.	Δx	slope	$y' = -.4$
	1/2	-.390159	
	1/4	-.389941	
	1/8	-.389887	
	1/16	-.389873	
	1/32	-.389870	

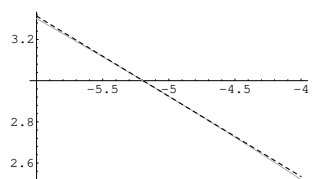
3. a. For $f(x) = \sin x$ at $x = 0$, the equation of the line is $y = x$. The y -intercept is 0 since $\sin(0) = 0$.



3. b. For $\varphi(t) = t + 2^{-t}$ at $t = 7$, using the value of the slope found in the earlier problem, the equation of the line is approximately $y = .99458t + c$. Since $\varphi(7) \approx 7.0078$, $c \approx .04574$, therefore $y \approx .99458t + .04574$.

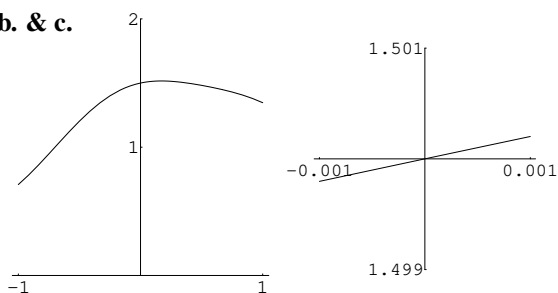


3. c. For $H(x) = x^{2/3}$ at $x = -5$, using the value of the slope found in the earlier problem, the equation of the line is approximately $y = -.38986x + c$. Since $H(-5) \approx 2.92402$, $c \approx .9747$, therefore $y \approx -.38986x + .9747$.



4. a. $f(0) = 3/2$.

4. b. & c.



4. d. $f(.001) = 1.5002020240590$ and $f(-.001) = 1.4997965581426$, so the slope is $.207329582$.

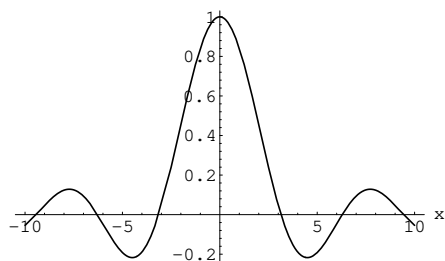
4. e. If we use a smaller $\Delta x = .0001$, we get

$$(f(.0001) - f(-.0001)) / .0002 \approx .207325581,$$

so we can say $f'(0) = .20732$, accurate to six decimal places.

4. f. The equation of the line in part (c) is approximately $y = .20732x + f(0) = .20732x + 1.5$.

5.



5. a. $g'(1) = -.301168\dots$, $g'(2.79) = -.380739\dots$, $g'(\pi) = -.318309\dots$, $g'(3.1) = -.326628\dots$

5. b. $g'(x) = 0$ for $x = 0, \pm 4.49341, \pm 7.2525$, for example.

5. c. The graph of g suggests that g' is most negative near $x = 2$. Through trial and error, zooming in on values of x near 2, we see g is decreasing most rapidly for $x \approx 2.08$ with $g'(2.08) \approx -.43618$.

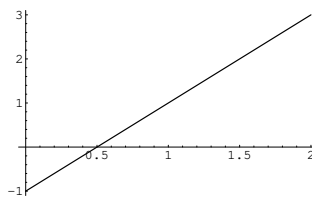
5. d. By trial and error, $g'(.80) \approx -.25$

5. e. $g(0) \approx 1$ and $g'(0) \approx 0$. Zooming in also suggests g is locally linear near $x = 0$.

5. f. $g(x) \rightarrow 1$ and $g'(x) \rightarrow 0$ as $x \rightarrow 0$.

Rates from Graphs; Graphs from Rates

6. a. If we look for a linear function, $f'(1) = 2$ implies that $y = 2x + c$, and $f(1) = 1$ implies $c = -1$. $y = 2x - 1$ satisfies the given conditions. In fact any other function which satisfies these conditions “shares phosphor” with this line at $(1, 1)$.



6. b. The function will have a slope of 2 at $(1, 1)$ and then a large negative slope to bring f down to -5 for $x = 1.1$.

7. a. A is in the lead during the first half of the race, and B is during the second half.

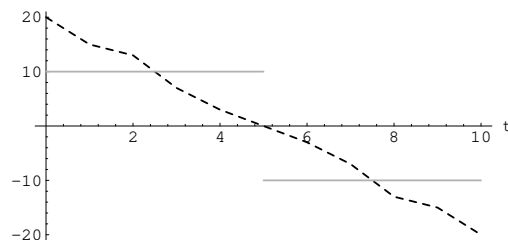
7. b. A appears to be the leader between the 2nd and 3rd seconds with a lead of about 12 feet. B appears to be the leader between the 7th and 8th seconds also with lead of about 12 feet.

7. c. A's velocity depends upon the interval used to compute the slope of the parabola. Taking into account values of distance only for $.5 \leq t \leq 1.5$, A's velocity is ≈ 15 feet/second after one second. B's velocity is a constant 10 feet/second in the first second.

7. d. B has constant velocity 10 feet/second during the first 5 seconds, and -10 feet/second during the second 5 seconds. A's velocity is constantly changing. For instance, A's velocity at time t can be found by calculating the slope of the line from $(t - .5, d_{t-.5})$ to $(t + .5, d_{t+.5})$. Using this method, we get the table below.

Elapsed time (sec)	Velocity (ft/sec)
1	15
2	13
3	7
4	3
5	0
6	-3
7	-7
8	-13
9	-15
10	-20

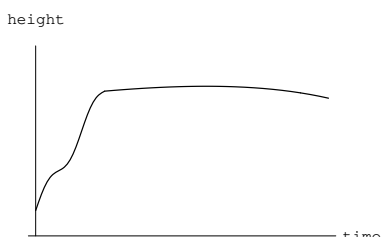
[Note: since calculated velocity varies with Δt and the interval of time chosen to be representative of t , variations are likely. Therefore the student should indicate how the calculations were done.]



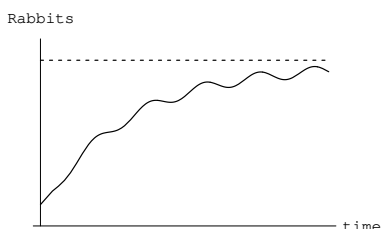
7. e. A goes faster than B during $0 \leq t \leq 2.5$ (approximately), as well as during $7.5 \leq t \leq 10$ seconds. B goes faster than A the remainder of the time: $2.5 \leq t \leq 7.5$. A is running -5 feet/second at $t \approx 6.5$ seconds.

8. The following graphs are possibilities that have the right qualitative features. Other graphs with the same qualitative features are equally acceptable.

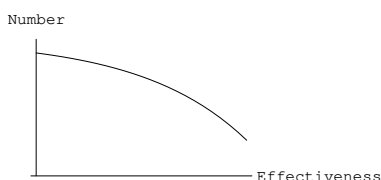
8. a.



8. b.



8. c.



9. a. The following table first estimates $f(a)$ = “distance traveled” from the graph, and then calculates the velocity as $(f(a+h) - f(a-h))/2h$, which is the pattern used in part iii of exercise 3 in section 3.1. For example, at time 2 the velocity is $(120 - 20)/2 = 50$.

elapsed time (secs)	distance (feet)	velocity (ft/sec)
1	20	30
2	60	50
3	120	70
4	200	90
5	300	105
6	410	115
7	530	125
8	660	135
9	800	145
10	950	150
11	1100	150
12	1250	150
13	1400	150

[Note: there are other ways to do this, and they may give different results. Students should indicate the methods they use.]

9. b. The velocity levels off at ≈ 150 feet/second.

9. c. If we use 150 feet/second as our final estimate of the parachutist’s velocity, we could expect the parachutist would fall an additional 300 feet from the 13th to the 15th second. Taking the distance fallen after 13 seconds as 1400 feet, the total after 15 seconds is then about 1700 feet.

10. a. False. Consider the function $g(t) = t - 250$, for example. This is a linear function with slope 1 (so $g'(t) > 0$ for all t), but $g(214) < 0$.

10. b. True. If $g'(t)$ is positive, then $g(t)$ is increasing. Thus if $y > x$ then $g(y) > g(x)$, so $g(214) > g(17)$.

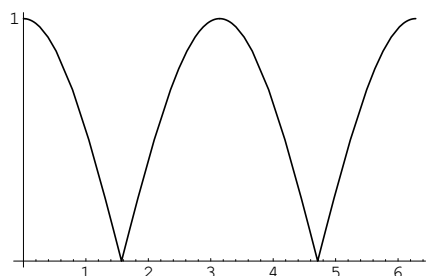
10. c. True. Since Samantha and Bill’s rates of speed are constant throughout the trip, Samantha is always 1 mile ahead, thus she will be 1 mile ahead at the end of the trip.

10. d. True. We need to show that the *difference* between Samantha’s and Bill’s speeds was zero at some instant. Since Samantha and Bill covered the same distance in the same amount of time, their average speeds were the same. This means that the average value of the difference in their speeds was zero. If the difference in speeds was *always* positive (or *always* negative), its average value couldn’t be zero. Therefore, the difference must have been positive sometimes and negative other times. (Actually, another possibility is that the difference was always zero—in which case there is nothing to prove!) Therefore, at some “intermediate” time, the difference must have been zero. [This is a difficult question!]

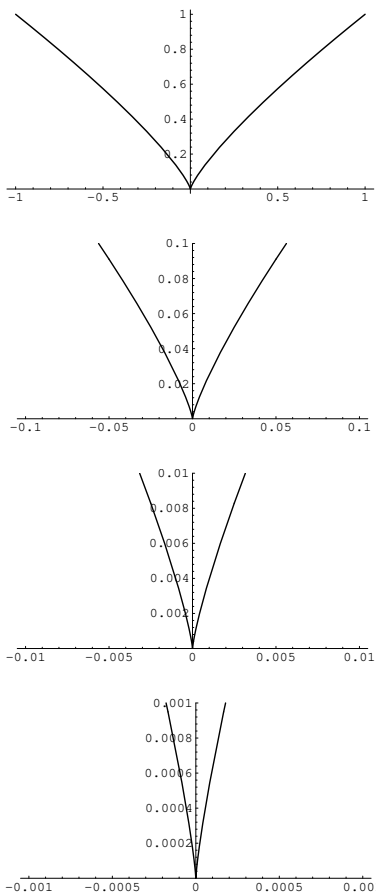
When Local Linearity Fails

11. As the window gets successively smaller, the “corner” at $x = 0$ continues to look the same.

12. The function is not locally linear at points x where $\cos x = 0$; that is, where $x = (2n + 1)\pi/2$, for n an integer.

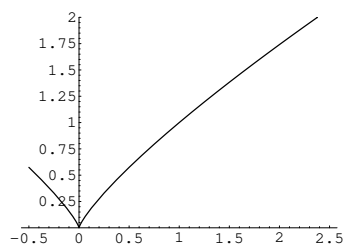


13.

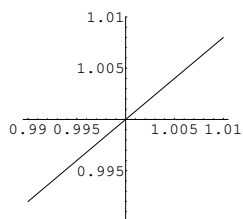


Under magnification, the corner at the origin gets more and more acute. This graph is not locally linear at $(0,0)$.

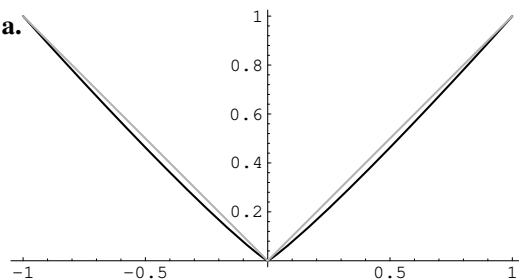
14.



The graph suggests $f(x) = x^{4/5}$ is locally linear at $x = 1$. This is confirmed by zooming in; the graph straightens out nicely.



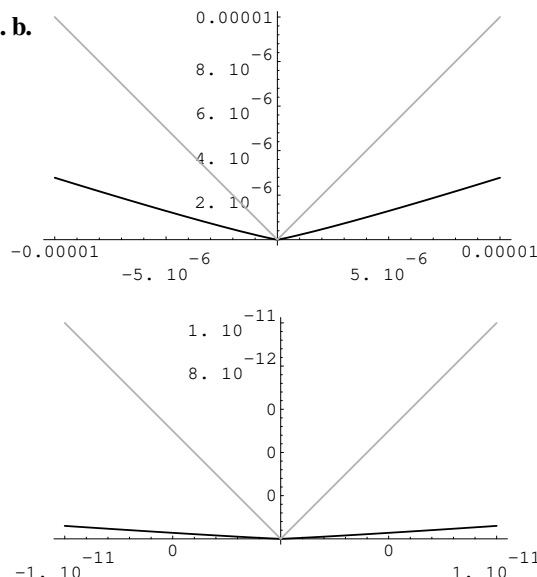
15. a.



[Note: $|x|$ is the grey graph, while $K(x)$ is the black graph.]

At this magnification, the function $K(x) = x^{10/9}$ appears to be similar to $|x|$: it looks like it has a corner at the origin.

15. b.

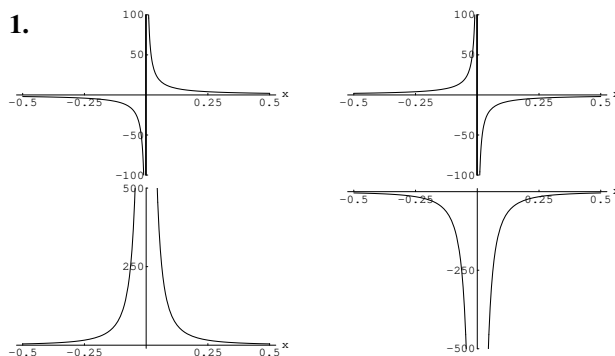


15. c. Under high magnification, the graph of K flattens out and becomes nearly horizontal near $x = 0$. Therefore, K is locally linear for all x ; it is *not* like $|x|$.

3.3 The Derivative

Computing Derivatives

1.



The function $y = -1/x$ is the only one that has a derivative which is always positive, since its graph is the only one whose slope is positive everywhere.

2.	$f = 1/x$	$f = \sin(7x)$
h	$f'(2)$	$f'(3)$
.1	-.25062657	-3.52856877
.01	-.25000625	-3.83097440
.001	-.25000006	-3.83407351
.0001	-.25000000	-3.83410451
.00001	-.25000000	-3.83410482
.000001	-.25000000	-3.83410482
h	$f = x^3$ $f'(200)$	$f = 2^x$ $f'(5)$
.1	120000.00999999	22.19847536
.01	120000.00009988	22.18088739
.001	120000.00000140	22.18071155
.0001	120000.00000000	22.18070980
.00001	120000.00001863	22.18070978
.000001	119999.99964610	22.18070978

To seven decimal places, the values are

$$f'(2) = -.2500000, \quad f'(200) = 120000.0000000,$$

$$f'(3) = -3.8341048, \quad f'(5) = 22.1807098.$$

3. a.	$h = 1/2^k$	Q_1	Q_2
	1.00000000	4.00000000	7.00000000
	0.50000000	3.25000000	4.75000000
	0.25000000	3.06250000	3.81250000
	0.12500000	3.01562500	3.39062500
	0.06250000	3.00390625	3.19140625
	0.03125000	3.00097656	3.09472656
	0.01562500	3.00024414	3.04711914
	0.00781250	3.00006104	3.02349854
	0.00390625	3.00001526	3.01173401

3. b. Q_1 stabilizes to four decimal places, while Q_2 only stabilizes to one decimal place.

3. c. Q_1 is a much better estimator. The best estimate of Q_2 gives $f' \approx 3.012$, while $h = 1/2^4$ which is 16 times as large as $h = 1/2^8$, gives $Q_1 \approx 3.004$.

4. a.	$h = 1/2^k$	Q_1	Q_2
	1.00000000	.16692527	.16227766
	0.50000000	.16673105	.16441400
	0.25000000	.16668275	.16552506
	0.12500000	.16667069	.16609195
	0.06250000	.16666767	.16637832
	0.03125000	.16666692	.16652224
	0.01562500	.16666673	.16659439
	0.00781250	.16666668	.16663051
	0.00390625	.16666667	.16664859

4. b. Q_1 stabilizes to (at least) seven decimal places, while Q_2 only stabilizes to the fourth decimal place.

4. c. Q_1 is a much better estimator. The best estimate of Q_2 gives $f' \approx .16665$, while a value of $h = 1/2^2$, which is 64 times as large as $h = 1/2^8$, already gives $Q_1 \approx .16668$.

5. Here is one possibility:

```
DEF fnf(x) = x ^ 3
a = 1
FOR k = 0 TO 8
  h = 1 / 2 ^ k
  q1 = (fnf(a + h) - fnf(a - h)) / (2 * h)
  q2 = (fnf(a + h) - fnf(a)) / h
  PRINT h, q1, q2
NEXT k
```

6. When $y = 2^x$, $dy/dx = .69314718\dots$ accurate to 8 decimal places. When $y = 3^x$, $dy/dx = 1.09861228\dots$ accurate to 8 decimal places. When $y = 10^x$, $dy/dx = 2.30258509\dots$ accurate to 8 decimal places. When $y = 1/2^x$, $dy/dx = -.69314718\dots$ accurate to 8 decimal places.

$$7. a. \frac{f(a+h) - f(a)}{h} = \frac{2^{a+h} - 2^a}{h} = 2^a \cdot \frac{2^h - 1}{h}.$$

$$7. b. \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} 2^a \cdot \frac{2^h - 1}{h} = 2^a \cdot \lim_{h \rightarrow 0} \frac{2^h - 2^0}{h} = 2^a \cdot f'(0).$$

Since 2^a doesn't change as $h \rightarrow 0$, we can move 2^a outside the limit.

$$8. a. \frac{f(x+h) - f(x)}{h} = \frac{b^{x+h} - b^x}{h} = b^x \cdot \frac{b^h - 1}{h}.$$

$$8. b. f'(0) = \lim_{h \rightarrow 0} \frac{b^h - 1}{h}. \text{ Therefore,}$$

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} b^x \cdot \frac{b^h - 1}{h} = b^x \cdot \lim_{h \rightarrow 0} \frac{b^h - 1}{h} = b^x \cdot f'(0).$$

9. a. $y = |x|$ is differentiable at each point where y is locally linear, that is, for all x except $x = 0$.

9. b. Since $|x| = -x$ when $x < 0$ and $|x| = x$ when $x > 0$, it follows that

$$\frac{dy}{dx} = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0. \end{cases}$$

The Microscope Equation

.

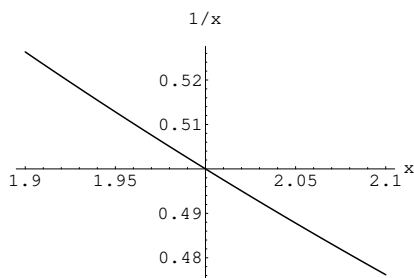
10. a. We use the value of the derivative as calculated in problem 2: $\Delta y \approx -.2500000 \Delta x$.

10. b. $\Delta y \approx -3.8341048 \Delta x$.

10. c. $\Delta y \approx 120000.0000000 \Delta x$.

10. d. $\Delta y \approx 22.1807098 \Delta x$.

11. a.



11. b. If $x = 2.05$, then $\Delta x = .05$ and $\Delta y \approx -.0125$. Therefore $f(2.05) \approx f(2) - .0125 = .4875$. The true value is $1/2.05 = .487804878\dots$, so the microscope approximation is about .0003 too small.

11. c. Here $\Delta x = .02$, so $\Delta y \approx -.25(.02) = .005$ and therefore $f(2.02) \approx .495$. The true value is $1/2.02 = .495049504\dots$, a difference of about .00005.

11. d. Here $\Delta x = -.005$ and $\Delta y \approx +.00125$. Thus $f(1.995) \approx .50125$. The true value is $1/1.995 = .501253132\dots$, a difference of about .000003.

12. a. If $T = 400/s$, then at 40 miles per hour the 400-mile trip takes 10 hours, $T(40)$.

12. b. The microscope equation for T is $\Delta T \approx -.25 \Delta s$.

12. c. As speed increases from 40 mph, travel time *decreases* at a rate of .25 hours/mph or 15 minutes/mph.

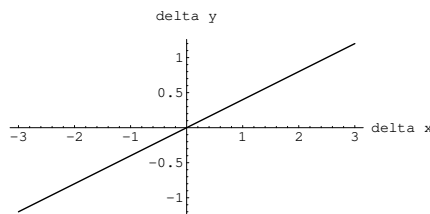
12. d. $\Delta T \approx .25(5) = 1.25$ hours, a savings of 1.25 hours.

12. e. $\Delta T \approx .16(-5) = -.8$ hours, a loss of .8 hours or 48 minutes.

12. f. No. The 40 mph microscope equation predicts a travel time of $10 - 1.25 = 8.75$ hours, while the 50 mph microscope equation predicts a travel time of $8 + .8 = 8.8$ hours.

13. a. $\Delta y \approx .4 \Delta x$ at $x = 5$.

13. b. No formula is needed for f . The origin in this graph, $(\Delta x, \Delta y) = (0, 0)$, is the same point as $(x, y) = (5, 12)$.

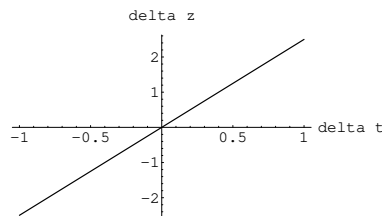


13. c. If $x = 5.3$, then $\Delta x = .3$ and $\Delta y \approx f'(5) \cdot \Delta x = .4(.3) = .12$, therefore $f(5.3) \approx 12.12$.

13. d. $f(5.23) \approx 12 + .4(.23) = 12.092$, $f(4.9) \approx 12 + .4(-.1) = 11.96$, $f(4.82) \approx 12 + .4(-.18) = 11.928$, $f(9) \approx 12 + .4(4) = 13.6$. The estimate for $f(9)$ is the least reliable, since $x = 9$ is farthest value for x from $x = 5$.

14. a. $\Delta z \approx 3.5 \Delta t$.

14. b. The origin in this graph, $(\Delta t, \Delta z) = (0, 0)$, is the same point as $(t, z) = (-4, 7)$.



14. c. $g(-4.2) \approx 7 + 3.5(-.2) = 6.3$, $g(-3.75) \approx 7 + 3.5(.25) = 7.875$.

14. d. For $g(t) = 6$, $\Delta z = 6 - 7 = -1$ and $\Delta t \approx \frac{2}{7} \Delta z = -\frac{2}{7}$. Thus $t \approx -4 - \frac{2}{7} = -4\frac{2}{7}$. For $g(t) = 8.5$, $\Delta z = +1.5$, $\Delta t \approx \frac{3}{7}$, and $t \approx -3\frac{4}{7}$.

15. The best estimate for $f(a+k)$ is $b - 3k$, since $f(a+k) \approx f(a) + f'(a) \cdot k = b - 3k$.

16. $\frac{f(a+h) - f(a-h)}{2h}$ and $\frac{f(a+2h) - f(a-h)}{3h}$ are reasonable estimates for $f'(a)$, since both quotients represent a change in the output, f , divided by the *corresponding* change in the input.

17. a. $D(5.1) \approx 30 + 5(.1) = 30.5$ feet,

$D(5.8) \approx 30 + 5(.8) = 34$ feet,

$D(4.7) \approx 30 + 5(-.3) = 28.5$ feet

17. b. $D'(3) \approx (26 - 22)/(3.1 - 2.8) = 4/.3 = 13\frac{1}{3}$ feet per second.

18. a. 2.8 **b)** 5.88 **c)** 1.8 **d)** 2.375 **e)** -1
f) -20

19. a. $P(1002) \approx \$504$, $P(995) \approx \$490$,
 $P(1006) \approx \$512$.

19. b. $P(2010) \approx \$2950$, $P(1992) \approx \$3040$,
 $P(1980) \approx \$3100$.

19. c. $P'(1234) \approx (634 - 625)/(1238 - 1234) = 9/4 =$
 $\$2.25/\text{unit}$.

3.4 Estimation and Error Analysis

Estimation

1. a. $T'(55) \approx (T(55.001) - T(45.999))/0.002 \approx$
 $-.03636363638$ and $T'(55) \approx (T(55.0001) -$
 $T(45.9999))/0.0002 \approx -.036363636368$, therefore
 $T'(55) = -.0363636363\dots$. As speed increases from 55
 mph, travel time *decreases* at a rate of .036 hours/mph or
 2 minutes/mph.

1. b. $\Delta T \approx -110\Delta v/v^2$ and $T = 110/v$ imply

$$\frac{\Delta T}{T} \approx \frac{-110/v^2}{110/v} \Delta v = -\frac{\Delta v}{v}.$$

We are given $\Delta v/v = 1\%$, so $\Delta T/T \approx -1\%$; your friend's
 driving time is 1% less than yours.

2. a. Since a cube has 6 congruent faces and we have 100
 ft² of plywood for each face, each edge is 10 ft wide and
 the volume is 1000 ft³.

2. b. If s is the length of a side, then $V = s^3$ and $A = 6s^2$.
 Consequently, $s = \sqrt{A/6}$ so

$$V(A) = \left(\sqrt{A/6}\right)^3 = (A/6)^{3/2} = \frac{A^{3/2}}{6\sqrt{6}}.$$

2. c. $V'(600) \approx (V(600.0001) - V(599.9999))/0.0002 \approx$
 2.5000000005 or simply 2.5. $V'(600)$ is the rate at which
 the volume of the box is growing, in cubic feet per square
 foot, when the surface area of the box is 600 square feet.

2. d. $\Delta V \approx 2.5\Delta A$, therefore $\Delta A \approx 10/2.5 = 4$ ft².

2. e. $\Delta A = -10$ ft² therefore $\Delta V \approx 2.5 \times -10 = -25$ ft³.
 The volume would decrease by about 25 ft³.

2. f. If 2% of the plywood is wasted then $\Delta A =$
 $-.02(600) = -12$ ft². $\Delta V = 2.5 \times 12 = -30$ ft³. $\Delta V/V =$
 $-30/1000 = -3\%$, a 3% decrease in volume.

3. Since $R(100) = .01$ and $R'(100) = -.0001$, $1/97 \approx$
 $R(100) + R'(100) \cdot \Delta s = .01 + (-.0001)(-3) = .0103$,
 $1/104 \approx .01 - .0004 = .0096$, and $R(94) \approx .0106$.

4. a. $f(4) = 2$ and $f'(4) = 1/4$, so $\sqrt{4.3} \approx 2 +$
 $(1/4)(.3) = 2.075$.

4. b. $\sqrt{5} \approx 2 + (1/4)(1) = 2.25$,
 $\sqrt{3.6} \approx 2 + (1/4)(-.4) = 1.9$

4. c. $f(100) = 10$ and $f'(100) = 1/20$; therefore
 $\sqrt{101} \approx 10 + (1/20)(1) = 10.05$,
 $\sqrt{99.73} \approx 10 + (1/20)(-.27) = 10 - .0135 = 9.9865$.

Error Analysis

5. a. The relative error is $\Delta L/L = .05/12.3 \approx .004$
 inch/inch or about 0.4%.

5. b. $A = s^2$. When $s = 12.3$ inches, $A = 12.3^2 =$
 151.29 inches². If we estimate $A'(12.3)$ by $(A(12.299) -$
 $A(12.301))/0.002 = 24.600000\dots$, then the error propaga-
 tion equation for the area is $\Delta A \approx 24.6\Delta s$ when $s = 12.3$
 inches. If the uncertainty in s is $\Delta s = \pm .05$ inches, then
 the uncertainty in A is $\Delta A \approx 24.6(\pm .05) = \pm 1.23$ in².

5. c. The relative error is $\Delta A/A = 1.23/151.29 \approx$
 $.0081$ in²/in² = 0.81%.

5. d. This question assumes we still start with $s = 12.3$
 inches, so the error propagation equation is still $\Delta A \approx$
 $24.6\Delta s$. If we want $\Delta A = \pm 1$ inch², then we need $\Delta s \approx$
 $\pm 1/24.6 = \pm .04$ inches. If we want $\Delta A = \pm .1$ inch², then
 $\Delta s \approx \pm .004$ inches.

6. a. The propagation equation for absolute error is $\Delta f \approx$
 $f'(x) \cdot \Delta x = 2x \cdot \Delta x$.

6. b. To get relative error, divide by f :

$$\frac{\Delta f}{f} = \frac{2x \cdot \Delta x}{x^2} = 2 \frac{\Delta x}{x}.$$

7. If the estimated time of the fall is 2.5 secs, the height of
 the building is $d = 16(2.5)^2 = 100$ feet. Also, $d'(2.5) \approx$
 $(d(2.5001) - d(2.4999))/0.0002 \approx 80$ so the error propa-
 gation equation for d is $\Delta d \approx 80\Delta t$. Since the uncer-
 tainty in calculating the time is $\Delta t = \pm .25$ sec, the esti-
 mated uncertainty in measuring the height of the building
 is $\Delta d \approx \pm 20$ feet.

8. a. Since $k = v(T)/\sqrt{T}$ is true for any temperature T ,
 we can take $T = 20^\circ\text{C} = 293^\circ\text{K}$ and obtain

$$k = 343.4/\sqrt{293} \approx 20.062 \text{ m/sec per (degree K)}^{1/2}.$$

[The units for k are unusual!]

8. b. Approximate $v'(293)$ by

$$20.062 \frac{\sqrt{293.0001} - \sqrt{292.9999}}{.0002} \approx .5860.$$

Then the error propagation equation is $\Delta v \approx .5860 \Delta T$. If the estimate in temperature is off by $\pm 5^\circ\text{K}$, then the estimate in the velocity is off by about $.5860 \times \pm 5 = \pm 2.930$ m/sec. This means the distance to the lightning flash is off by

$$5 \text{ sec} \times \pm 2.930 \text{ m/sec} = \pm 14.65 \text{ meters.}$$

It seems the calculated distance is *not* very sensitive to an uncertainty in the temperature.

By contrast, an error of .25 seconds in measuring the elapsed time between the observation of the lightning flash and the sound would result in an error of $.25 \text{ sec} \times 343.4 \text{ m/sec} = 85.85$ meters in measuring the distance to the lightning strike. We notice that each degree of error in temperature measurement results in a .586 m/sec error in measuring velocity for an error of about 14.65 meters over a 5-second interval, while each error of as little as .1 second in measuring the time causes an error of 34 meters in measuring distance to the strike. This is more than twice as much as the error caused by a 5 degree Kelvin (or Celsius) temperature error. Thus temperature measurement as a source of error is far less significant than imprecision in the time measurement.

9. We measure the time t (in seconds) for a laser beam to take a *round-trip* to the moon, and calculate the distance the beam travels as $D = 300000t$ km or $3 \times 10^8 \times t$ meters. The distance to the moon is then $D/2$. The uncertainty in D is

$$\Delta D = (3 \times 10^8)t \text{ meters.}$$

(We can use an equals sign because D is a linear function of t and $D' = 3 \times 10^8$ m/sec. Another consequence of linearity is that the error propagation equation is independent of t and D .)

If we want $D/2$ to be certain to within $\pm .1$ meter, then D itself must be certain to within $\pm .2$ meter and therefore t must be known to within

$$t = \frac{\pm .2 \text{ meter}}{3 \times 10^8 \text{ m/sec}} \approx 6.7 \times 10^{-10} \text{ sec.}$$

3.5 A Global View

Sketching the Graph of the Derivative

1. *Ad libitum*. There are many possibilities, but they consist of pairs of functions whose graphs are **vertical translates** of each other. In particular, two linear functions with the same slope will provide such an example.

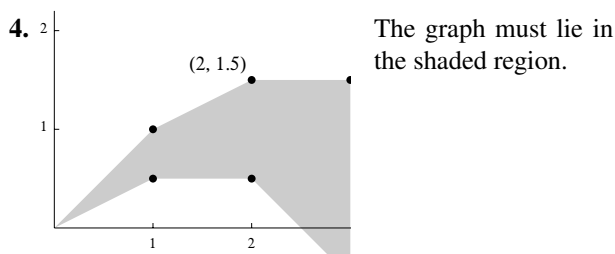
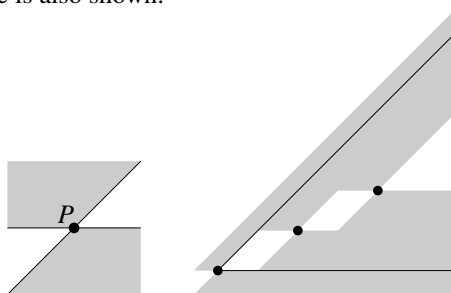
2. a. $s(t)$ b. $c(t) = s(2t)$ c. $s'(t)$ d. $c'(t)$

3. a. We restrict x to the interval $0 \leq x \leq 3$. Since $g'(x) \geq 0$ on the interval, g is increasing (or at least not

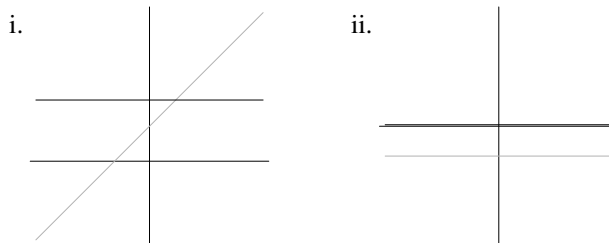
decreasing). Since $g(0) = 0$, $g(x) \geq 0$ for every x . Thus the graph of g must lie on or above the line $y = 0$ which forms the bottom of the triangle in the figure.

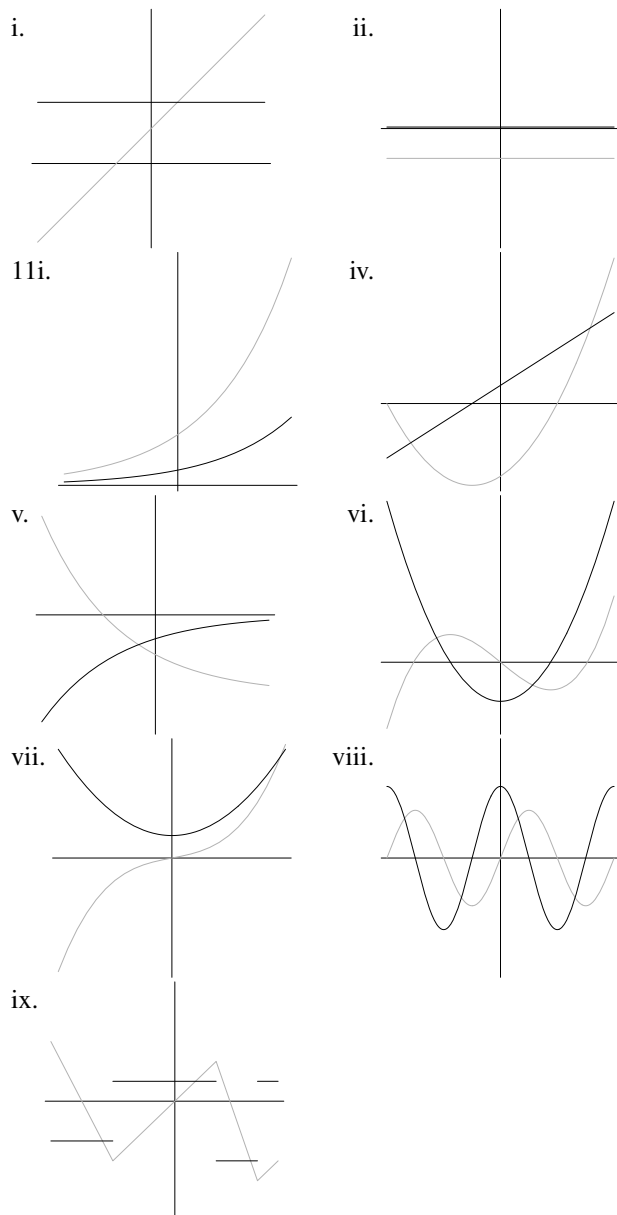
Next consider the function $G(x) = g(x) - x$. Since $g'(x) \leq 1$, $G'(x) = g'(x) - 1 \leq 0$, so G is decreasing. Since $G(0) = g(0) - 0 = 0$, it must be true that $G(x) \leq 0$ for every x . This means $g(x) \leq x$ for every x . Thus the graph of g must lie on or below the line $y = x$ which forms the top of the triangle in the figure.

3. b. If the graph of g passes through the point P (as in the figure on the left, below) and we know $0 \leq g'(x) \leq 1$ for all x , then the graph cannot enter the shaded region (which extends to infinity). This is essentially what the argument in part (a) says when it is continued to include points to the left of P . To answer part (b), use this pattern as a “mask” at each of the points $(0, 0)$ (already done), $(1, .5)$ and $(2, 1)$. In the figure on the right, these masks eliminate all but the unshaded points: the graph of g must stay in the unshaded region shown. For reference, the original triangle is also shown.



5. In each of the following sketches, the original function (in gray) and its derivative (in black) are shown on the same coordinate plane. Only qualitative features of the derivative are significant here.





Note: there are three points in ix. where the derivative is undefined.

Differentiation

6. a. $21x^6 - 1.2x^3 + 3\pi x^2$.

6. b. $\frac{1}{2}\sqrt{3}x^{-1/2} - 35x^{-6} = \frac{\sqrt{3}}{2\sqrt{x}} - \frac{35}{x^6}$.

6. c. $16w^7 - \cos w - \frac{2}{3w^3}$.

6. d. $-4\sin u - 3\sec^2 u + \frac{1}{3u^{2/3}}$

6. e. $-\frac{1}{4s^{3/4}}$

6. f. $\sqrt{7} \cdot k_2 \cdot 2^z + k_{1/2} \frac{1}{2^z}$.

Note: some students may be aware of the fact that the “fudge factors” k_2 and $k_{1/2}$ are $\ln 2 = .69\dots$ and $\ln(1/2) = -\ln 2 = -.69\dots$, respectively.

6. g. $-at + v_0$

7. In each case the graphs of g and $h = f'$ should coincide. The derivatives are:

7. a. $f'(x) = 4x^3$

7. b. $f'(x) = -x^{-2}$

7. c. $f'(x) = \frac{1}{2\sqrt{x}}$

7. d. $f'(x) = \cos x$

8. An arbitrary constant may (*but need not be*) added to any of these answers.

8. a. x^{12}

8. b. $\frac{5}{8}x^8$

8. c. $\sin x - \cos x$

8. d. $\frac{ax^3}{3} + \frac{bx^2}{2} + cx$

8. e. 0

8. f. $10\sqrt{x}$

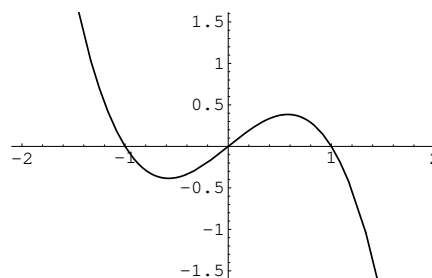
9. $y' = 1 - \frac{1}{2\sqrt{x}}$, so $y'(4) = 3/4$. $y'(100) = 19/20 = .95$. $y'(10000) = 199/200 = .995$.

10. a. $x - x^3$ is increasing when its derivative is positive. This occurs when $-\sqrt{1/3} < x < \sqrt{1/3}$, since the derivative of this function is $1 - 3x^2$.

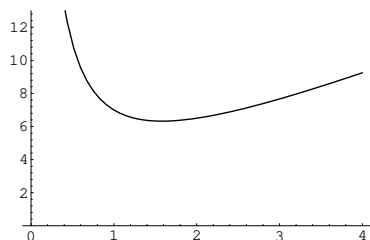
10. b. The graph of $y = x - x^3$ is rising most steeply at the point when its derivative is largest. The derivative is $1 - 3x^2$, and this is largest when $x = 0$.

10. c. The graph of $y = x - x^3$ is horizontal when its derivative (slope) is 0. This occurs when $x = \pm\sqrt{1/3}$.

10. d.



11. a.



11. b. As stated in the text, the lowest point on the graph occurs when the derivative $y' = 2 - \frac{5}{x^2}$ is 0, that is, when $x = \sqrt{2.5}$.

12. The derivative of $y = \sin x + \cos x$ is given by $y' = \cos x - \sin x$, so $y'(\pi/4) = 0$. Thus the slope of the graph is 0 at $x = \pi/4$.

13. a. The microscope equation for $y = \sin x$ at $x = 0$ is given by $\Delta y = \Delta x$, since the derivative $y' = \cos x$ is equal to 1 if $x = 0$.

13. b.	x	microscope	calculator
	.3	.3	.29552021
	.007	.007	.0069999428
	-.02	-.02	-.019998667

14. a. The microscope equation for $y = \tan x$ at $x = 0$ is given by $\Delta y = \Delta x$, since the derivative $y' = \sec^2 x$ is equal to 1 if $x = 0$.

14. b.	x	microscope	calculator
	.3	.3	.309336249
	.007	.007	.0070001143
	-.02	-.02	-.020002667

15. a. $\Delta y = \frac{1}{120} \Delta x$, for x near 3600.

15. b. $\sqrt{3628} \approx \sqrt{3600} + \frac{1}{120}(28) = 60.23\bar{3}$.

Calculator: $\sqrt{3628} = 60.232881$

Difference is about 0.00045.

$\sqrt{3592} \approx \sqrt{3600} + \frac{1}{120}(-8) = 59.93\bar{3}$.

Calculator: $\sqrt{3592} = 59.933296$.

Difference is about 0.00037.

16. a. $V' = 4\pi r^2$; when $r = 4$ inches, V is growing at the rate of $V' = 64\pi \approx 201$ cubic inches per inch (of growth in r).

16. b. $\Delta V \approx 64\pi \Delta r \approx 201 \Delta r$.

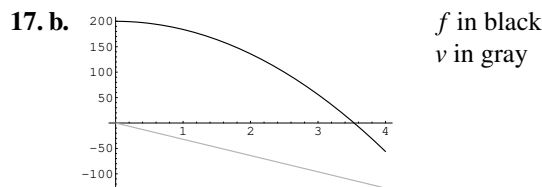
16. c. If $\Delta V = 50$ cubic inches, then $\Delta r \approx 50/201 = .249$ inches.

16. d. We have $\Delta V / \Delta r = 10$ cubic inches per second. We can find $\Delta r / \Delta t$ by dividing the microscope equation by Δt :

$$10 = \frac{\Delta V}{\Delta t} \approx 201 \frac{\Delta r}{\Delta t}.$$

Thus r is increasing at the rate of $\Delta r / \Delta t \approx 10/201 \approx .050$ inches per second.

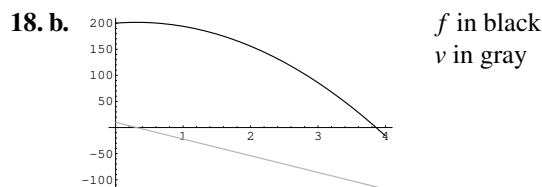
17. a. $v = f'(t) = -32t$ feet per second. When $t = 0$, then $v = 0$, as required.



The time interval $0 \leq t \leq 3.54$ makes sense, because $f(t) \geq 0$ then.

17. c. The ball hits the ground when $d = -16t^2 + 200 = 0$. Thus $t = \sqrt{200/16} \approx 3.536$ seconds. At this point, $v = -113.152$ feet per second.

18. a. $v = f'(t) = -32t + 10$. This is very similar to the function for velocity in the previous exercise, $v = f'(t) = -32t$. In both cases the velocity is a linear function of time with slope -32 (acceleration due to gravity). The only difference is in the v -intercepts for the lines, that is, in the initial velocity.



18. c. The ball rises when $v(t) = -32t + 10 > 0$, that is, when $t < 10/32 \approx .3125$ sec (and $t > 0$). It falls during the interval $.3125 < t < 3.861$. It reaches its highest point when $t = .3125$.

18. d. The ball rises to a height of $f(.3125) = 201.5625$ feet.

19. $v(t) = -32t - 40$.

19. a. The distance is given by $d = -16t^2 - 40t + 200$. Since $v = -40$ when $t = 0$, the initial condition is satisfied.

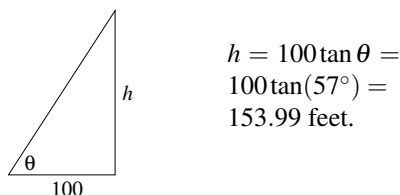
19. b. The ball hits the ground when $0 = d = -16t^2 - 40t + 200$. Since d can be factored as $-8(2t - 5)(t + 5)$, we see $d = 0$ when $2t - 5 = 0$; that is, when $t = 2.5$ secs. Its velocity is $-32 \times 2.5 - 40 = -120$ ft/sec at that time.

20. a. The velocity is given by $v = d'(t) = 3 \cos t$. When the velocity is positive, the ball is moving toward the right; when the velocity is negative, the ball is moving toward the left; when the velocity is 0, the ball is pausing momentarily (as it changes direction, i.e., changing from moving right to moving left and vice-versa).

20. b. The ball never gets farther than 3 units from the center of the track since the maximum value of $3 \sin t$ is 3, and the minimum value is -3 .

20. c. When the ball is at the midpoint of the track its velocity is either 3 in/sec (moving right) or -3 in/sec (moving left). It never goes faster than this since the maximum value of the velocity $v = 3 \cos t$ is 3, and the minimum value is -3 .

21. a.



21. b. Convert to radians: $57^\circ = 57 \cdot \pi/180 = .995$ radians and $5^\circ = 5 \cdot \pi/180 = .087$ radians. The error propagation equation for the equation $h = 100 \tan \theta$ at $\theta = .995$ radians says:

$$\Delta h \approx 100 \sec^2(.995) \Delta \theta \approx 337.3 \Delta \theta.$$

If the measurement of θ is certain only to $\Delta \theta = .087$ radians, then the uncertainty in the calculated value of h is $\Delta h \approx 337.3 \times .087 = 29.34$ feet.

22. For any angle θ (measured in radians), the percentage error in h is

$$\frac{\Delta h}{h} \approx \frac{100 \sec^2(\theta) \Delta \theta}{100 \tan(\theta)} = \frac{\Delta \theta}{\sin(\theta) \cos(\theta)} = \frac{2 \Delta \theta}{\sin(2\theta)}.$$

The last expression follows from the trig identity $\sin(\theta) \cos(\theta) = \frac{1}{2} \sin(2\theta)$.

22. a. Taking $2\theta = 2 \times 57^\circ = 1.99$ radians and $\Delta \theta = 1^\circ = .0175$ radian, we have

$$\frac{\Delta h}{h} \approx \frac{2 \times .0175}{\sin(1.99)} \approx .038.$$

Thus a 1° error in the measurement of θ propagates to a 3.8% error in the calculated height.

22. b. If we change θ to $75^\circ = 5\pi/12$ radians, then $\sin(2\theta) = \sin(5\pi/6) = \sin(\pi/6) = 1/2$ (exactly!). The percentage equation therefore takes the extremely simple form

$$\frac{\Delta h}{h} \approx \frac{2 \Delta \theta}{1/2} = 4 \Delta \theta \approx 4 \times .0175 = .07.$$

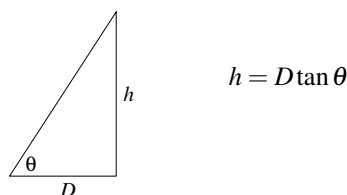
This is a 7% error in h , when $\theta = 75^\circ$.

If we take $\theta = 40^\circ = .698$ radians, then

$$\frac{\Delta h}{h} \approx \frac{2 \times .0175}{\sin(1.396)} \approx .0355$$

This is a 3.55% error in h , when $\theta = 40^\circ$.

22. c.



We have a new variable: D , the distance from the viewer to the tree. D is related to θ by $h = D \tan(\theta)$. Therefore $\Delta h \approx D \sec^2(\theta) \Delta \theta$ and

$$\frac{\Delta h}{h} \approx \frac{D \sec^2(\theta) \Delta \theta}{D \tan(\theta)} = \frac{\Delta \theta}{\sin(\theta) \cos(\theta)} = \frac{2 \Delta \theta}{\sin(2\theta)}.$$

Our aim is to choose D so that we can calculate h as accurately as possible. We take this to mean that we want Δh as small as possible.

Solution 1. Since D and θ are related (by $h = D \tan(\theta)$), we can look for the θ that minimizes Δh in the equation

$$\frac{\Delta h}{h} \approx \frac{2 \Delta \theta}{\sin(2\theta)}.$$

Now h and $\Delta \theta$ are fixed (at 150 feet and 1° , respectively). Therefore, Δh will be minimized when $\sin(2\theta)$ is maximized. This happens when $2\theta = \pi/2$, or $\theta = \pi/4$ (or 45°). Hence $D = h / \tan(45^\circ) = h$.

Thus, if the tree is about 150 feet tall, you should stand about 150 feet away from its base to measure the angle θ . The uncertainty in the height will then be

$$\Delta h \approx 2 \times 150 \times .0175 = 5.25 \text{ feet.}$$

Solution 2. We work directly with the variable D . Since $\tan(\theta) = h/D$ and $\sec^2(\theta) = 1 + \tan^2(\theta)$, the equation $\Delta h \approx D \sec^2(\theta) \Delta \theta$ becomes

$$\Delta h \approx D \left(1 + \frac{h^2}{D^2} \right) \Delta \theta = \left(D + \frac{h^2}{D} \right) \Delta \theta.$$

Since $\Delta \theta$ is fixed, we can minimize Δh by setting the derivative of Δh with respect to D equal to 0. Now

$$\frac{\partial(\Delta h)}{\partial D} \approx \left(1 - \frac{h^2}{D^2} \right) \Delta \theta,$$

so $\partial(\Delta h)/\partial D \approx 0$ when $1 - h^2/D^2 = 0$, i.e., when $D = h$. This is the same conclusion we reached before.

3.6 The Chain Rule

1. a. $dy/du = 5$ and $du/dx = -7$, so $dy/dx = 5(-7) = -35$.

1. b. $dy/du = \cos u$ and $du/dx = -7$, so $dy/dx = \cos u \cdot (-7) = -7\cos(4-7x)$.

1. c. $dy/du = \sec^2 u$ and $du/dx = 3x^2$, so $dy/dx = (\sec^2 u)(3x^2) = 3x^2 \sec^2(x^3)$.

1. d. $dy/du = k_{10}10^u$ and $du/dx = 2x$, so $dy/dx = 2x \cdot k_{10}10^{x^2}$. (Note: $k_{10} \approx 2.3$ so $dy/dx \approx 4.6x10^{x^2}$.)

1. e. $dy/du = 4u^3$ and $du/dx = 3x^2$, so $dy/dx = 12x^2(x^3+5)^2$.

2. a. Let $y = u^5$ where $u = 9x + 6x^3$. Then $F'(x) = dy/dx = 5u^4(9 + 18x^2) = 5(9x + 6x^3)^4(9 + 18x^2)$.

2. b. Let $y = \sqrt{u}$ where $u = 4w^2 + 1$. Then $dy/du = 1/2\sqrt{u}$ and $du/dw = 8w$, so $G'(w) = dy/dw = 4w/\sqrt{4w^2 + 1}$.

2. c. Let $y = \sqrt{u}$ where $u = b^3$ and $b = 4w^2 + 1$. Then $dy/du = 1/2\sqrt{u}$, $du/db = 3b^2$, and $db/dw = 8w$, so $S'(w) = dy/dw = (dy/du)(du/db)(db/dw) = 24w(4w^2 + 1)^2/2\sqrt{(4w^2 + 1)^3} = 12w(4w^2 + 1)/\sqrt{4w^2 + 1}$.

2. d. Let $y = u^{-1}$ where $u = 1 - x$. Then $dy/du = -1/u^2$, $du/dx = -1$, and $R'(x) = dy/dx = 1/(1-x)^2$.

2. e. Let $y = 3\tan u$ where $u = 1/z$. Then $dy/du = 3\sec^2 u$, $du/dz = -z^{-2}$, and $D'(z) = dy/dz = -3z^{-2}\sec^2(1/z)$.

2. f. Let $y = u^2$ where $u = \sin r$ and $r = w^3 + 1$. Then $dy/du = 2u$, $du/dr = \cos r$, and $dr/dw = 3w^2$, so $dy/dw = 2u(\cos r)(3w^2) = 6w^2 \sin(w^3 + 1) \cos(w^3 + 1)$. Thus,

$$\text{dog}'(x) = 6w^2 \sin(w^3 + 1) \cos(w^3 + 1).$$

2. g. Let $y = \cos u$ where $u = 2^t$. Then $dy/du = -\sin u$ and $du/dt = k_2 2^t$, so

$$\text{pig}'(t) = dy/dt = -k_2 2^t \sin 2^t$$

where $k_2 \approx .69$.

2. h. Let $y = 5^u$ where $u = 1/x$. Then $dy/du = k_5 5^u$ and $du/dx = -1/x^2$, so

$$\text{wombat}'(x) = dy/dx = -k_5 5^{1/x}/x^2$$

where $k_5 \approx 1.60$.

3. Since $h'(x) = 6(f(x))^5 \cdot f'(x)$, if $f(93) = 2$ and $f'(93) = -4$, then $h'(93) = 6 \times 2^5 \times -4 = -768$.

4. We know $H'(x) = F'(x^2 - 4x + 2) \cdot (2x - 4)$. Therefore, $H'(4) = F'(4 - 8 + 2) \cdot (8 - 4) = 4F'(2) = 6$, because $F'(2) = 3$ is given.

5. If $y = u^5$ and $u = 1 + x^2$, then $f'(x) = dy/dx = 5(1 + x^2)^4 \cdot 2x = 10x(1 + x^2)^4$. Therefore $f'(0) = 0$ and $f'(1) = 160$.

6. If $y = \cos u$ and $u = \sin t$, then $h'(t) = dy/dt = -\sin u \cos t = -\sin(\sin t) \cos t$, therefore, $h'(0) = 0$ and $h'(\pi) = 0$.

7.

function	derivative
$f(x+17)$	$f'(x+17)$
$f(17x)$	$17f'(17x)$
$17f(x)$	$17f'(x)$
$17+f(x)$	$f'(x)$
$f(17)$	0

Thus only $17 + f(x)$ has $g(t) = f'(t)$ as its derivative.

8. $f'(t) = 2t + 2$.

$$g'(t) = 15t^2.$$

$$g(f(t)) = 5(t^2 + 2t)^3 - 3 = -3 + 40t^3 + 60t^4 + 30t^5 + 5t^6.$$

$$f(g(t)) = (5t^3 - 3)^2 + 2(5t^3 - 3) = 3 - 20t^3 + 25t^6.$$

$$g'(f(t)) = 15(t^2 + 2t)^2 = 60t^2 + 60t^3 + 15t^4.$$

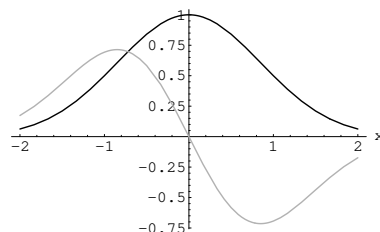
$$f'(g(t)) = 2(5t^3 - 3) + 2 = -4 + 10t^3.$$

$$(f(g(t)))' = -60t^2 + 150t^5.$$

$$(g(f(t)))' = 120t^2 + 240t^3 + 150t^4 + 30t^5.$$

9. a. $f'(x) = -2xk_2 2^{-x^2}$.

9. b.

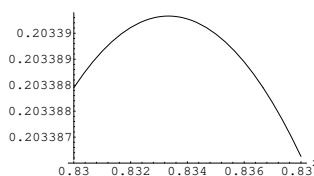


f' is the gray graph.

9. c. $f'(x) = 0$ only when $x = 0$. The graph of f is horizontal at this point; it has a maximum there.

9. d. f has a positive slope for $x < 0$ and a negative slope for $x > 0$, as f' is above the x -axis for $x < 0$ and below the x -axis for $x > 0$.

10. a.



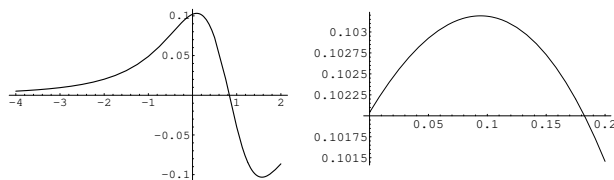
The axes origin has been shifted to $(.830, .203387)$.

From the graph it appears the maximum occurs where $x = .83$ (to 2 decimal places accuracy).

10. b. $y' = -(6x - 5)(3x^2 - 5x + 7)^{-2}$, and $y' = 0$ where the numerator $6x - 5$ equals 0.

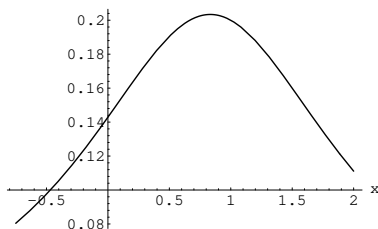
10. c. y takes its maximum where $x = 5/6 = .83\bar{3}$.

10. d. y will be rising most steeply at the point where y' is largest. To find where y' is largest, inspect the graph of y' .



It appears that y' is largest (and hence y is steepest) when $x \approx .09$. Note: since students have not yet been exposed to the product and quotient rules for derivatives, they cannot be expected to find the maximum of y' by taking its derivative.

Here is a fuller view of the graph of y that shows it is steepest near $x \approx .1$.



11. a. $y' = \cos \sqrt{x}/2\sqrt{x}$ and $y'(1) = \cos(1)/2 \approx .27015115$. The microscope equation is $\Delta y \approx .27015115 \Delta x$.

11. b. We start with $\sin 1 = .84147098$ (from a calculator). Then $\sin \sqrt{1.05} \approx .84147098 + .20715115(.05) = .85497854$.

Directly by calculator, $\sin \sqrt{1.05} = .85455586$.

$\sin \sqrt{.9} \approx .84147098 + .20715115(-.1) = .81445587$.

Directly by calculator, $\sin \sqrt{.9} = .8126489$.

12. a. $w' = 1/2\sqrt{1+x}$, so $w'(0) = 1/2$. The microscope equation is $\Delta w \approx (1/2) \Delta x$.

12. b. $\sqrt{1.1056} \approx 1 + (1/2)(.1056) = 1.0528$.

By calculator, $\sqrt{1.1056} \approx 1.05148$, a difference of less than .002.

$\sqrt{.9788} \approx 1 + (1/2)(-.0212) = .9894$.

By calculator, $\sqrt{.9788} \approx .989343$, a difference of less than .00006.

13. The surface area A is a function of the length s of the side. In particular, $dA/ds = 60 \text{ in}^2/\text{in}$. Furthermore, s depends on the time t . In particular, $ds/dt = 3 \text{ in/hr}$. By the chain rule, A also depends on t and $dA/dt = dA/ds \cdot ds/dt = 60 \times 3 = 180 \text{ in}^2/\text{hr}$.

14. $f(x) = (5+x^3)^{11}/11 + C$,
 $p(x) = (5+x^3)^{11}/33 + C$; C can be any constant.

15. $g(t) = \sqrt{1+t^2}$.

3.7 Partial Derivatives

1. a. $f_x = 2xy$, $f_y = x^2$.

1. b. $f_x = 1/2\sqrt{x+y}$, $f_y = 1/2\sqrt{x+y}$.

1. c. $f_x = 2xy + 15x^2 - 1/2\sqrt{x+y}$,
 $f_y = x^2 - 1/2\sqrt{x+y}$.

1. d. $f_x = k_{10}y10^{xy}$, $f_y = k_{10}x10^{xy}$ where $k_{10} \approx 2.3026$.

1. e. $f_x = -y/x^2$, $f_y = 1/x$.

1. f. Let $f = \sin u$ and $u = x/y$. Then $\partial f/\partial x = df/du \cdot \partial u/\partial x$. Since $df/du = \cos u$,
 $f_x = -\cos(y/x)y x^{-2}$, $f_y = \cos(y/x)/x$.

1. g. $f_x = 34xy^{-3} - 2x \sin y$, $f_y = -51x^2y^{-4} - x^2 \cos y$.

1. h. $f_u = v/5 - 5/vu^2$, $f_v = u/5 - 5/uv^2$.

1. i. $f_x = (\sqrt[3]{y}/\sqrt{x}) + 7 \sin x$, $f_y = 2\sqrt{x}/3\sqrt[3]{y^2}$.

1. j. $f_x = \tan y$, $f_y = x \sec^2 y$.

2. March 7 is day 64 and March 5 is day 62. The basic microscope equation to help us here is

$$\Delta T \approx T_d(64, 40^\circ\text{N}) \cdot \Delta d + T_\lambda(64, 40^\circ\text{N}) \cdot \Delta \lambda.$$

We estimate the partial derivatives by evaluating T on either side of $(d, \lambda) = (64, 40^\circ\text{N})$.

$$T_\lambda : \frac{6 : 26 - 6 : 22}{4^\circ} = 1 \text{ min/deg}$$

$$T_d : \frac{6 : 19 - 6 : 31}{70 - 62} = \frac{-12}{8} = -1.5 \text{ min/day}$$

Therefore, filling these values, $\Delta d = -2$, and $\Delta T = 0$ into the microscope equation, we get

$$0 = (-1.5 \times -2) + (1 \times \Delta \lambda).$$

Therefore $\Delta \lambda = -3$ deg. Since $\lambda = 40^\circ\text{N}$, this means that, at latitude 37°N , the sun rises at 6:25 on March 5.

3. a. $V_T = R/P$, $V_P = -RT/P^2$.

3. b. The unit for RT/P are

$$\frac{\text{newton-meters}}{^\circ\text{K}} \frac{^\circ\text{K}}{\text{newtons/meter}^2} = \text{meter}^3.$$

Since these are units for V , the units of the ideal gas law are consistent.

3. c. The microscope equation is

$$\Delta V \approx \frac{R}{P} \Delta T - \frac{RT}{P^2} \Delta P.$$

When $T = 350^\circ\text{K}$ and $P = 20 \text{ n/m}^2$, then $R/P = 8.33 \times 10^3/20 = 416.5$ and $-RT/P^2 = -8.3 \times 10^3(350)(20^{-2}) = -7262.5$. Since we also know $\Delta T = 10$ and $\Delta V = 1$, the microscope equation becomes

$$1 \approx (416.5 \times 10) + (-7262.5 \Delta P).$$

Hence $\Delta P \approx 4165/7262.5 \approx .57$ newtons/m². The pressure will increase.

4. The partial derivatives are the coefficients of the linear function. Thus

$$F(x, y) = .15x + 2.31y + F(0, 0).$$

Since $F(0, 0) = 8 - .15(4) - 2.31(1)$, it follows that

$$F(x, y) = .15x + 2.31y + 5.09.$$

5. a. $F_x = m, F_y = n$.

$$\text{5. b. } \frac{F(a + \Delta x, b) - F(a, b)}{\Delta x} = \frac{m(a + \Delta x) + nb + c - (ma + nb + c)}{\Delta x} = \frac{m\Delta x}{\Delta x} = m.$$

$$\text{5. c. } F(a + \Delta x, b + \Delta y) - F(a, b) = m(a + \Delta x) + n(b + \Delta y) + c - (ma + nb + c) = m\Delta x + n\Delta y.$$

6. a. Since G is a linear function of u , $\Delta G/\Delta u = (G(1 + \Delta u, 2) - G(1 - \Delta u, 2))/2\Delta u$ has the value $2/5 = .4$ for every value of Δu .

$$\text{6. b. } \Delta G/\Delta v \approx (G(1, 2 + \Delta v) - G(1, 2 - \Delta v))/2\Delta v$$

Δv	$\Delta G/\Delta v$
$\pm .1$	.12004802
$\pm .01$	.12000048
$\pm .001$	.12000000
$\pm .0001$	.12000000
$\pm .00001$	.12000000

6. c. Since $G_u(1, 2) = .4$ and $G_v(1, 2) \approx .12$ the microscope equation is

$$\Delta G \approx .4\Delta u + .12\Delta v.$$

6. d. $\Delta u = -.2$ and $\Delta v = .1$, so

$$\Delta G \approx (.4 \times -.2) + (.12 \times .1) = -.068.$$

Thus $G(.8, 2.1) \approx G(1, 2) + \Delta G = \frac{2}{5} - .068 = .332$. The exact value is $G(.8, 2.1) = .32941 \dots$, a difference of less than .0026.

$$\text{7. a. } A = 51 \times 2034 = 103734 \text{ ft}^2.$$

7. b. Let $A = n \cdot l$, where n is the length of the narrow piece and l is the length of the long piece. The error in A is given by the microscope equation:

$$\Delta A \approx A_n \Delta n + A_l \Delta l.$$

Since $A_n = l, A_l = n$, we have

$$\Delta A = l \cdot \Delta n + n \cdot \Delta l = (2034 \times 1/3) + (51 \times 10) = 1188 \text{ ft}^2.$$

The percentage error is $1188/103734 \approx .0115 \approx 1.2\%$.

8. a. The microscope equation is

$$\Delta z \approx f_x(3, 12)\Delta x + f_y(3, 12)\Delta y = 7\Delta x + 4\Delta y.$$

Since $f(3, 12) = 240$, we get

$$f(4, 12) \approx 240 + (7 \times 1) + (4 \times 0) = 247;$$

$$f(3, 13) \approx 240 + (7 \times 0) + (4 \times 1) = 244;$$

$$f(4, 13) \approx 240 + (7 \times 1) + (4 \times 1) = 251;$$

$$f(4, 10) \approx 240 + (7 \times 1) + (4 \times -2) = 239.$$

8. b. $\Delta x = (.01 \times 3) = .03$ so $\Delta z \approx (7 \times .03) = .21$ while $\Delta y = (.01 \times 12) = .12$ so $\Delta z \approx (4 \times .12) = .48$. Therefore a 1% increase in y has the larger effect on z .

9. a. The microscope equation is

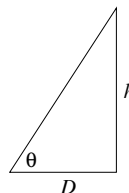
$$\Delta P \approx -.12\Delta K - .2\Delta L.$$

If expenses increase to \$24,000, then $\Delta K = .5$ so $\Delta P \approx (-.12 \times .5) = -.06$, a decrease in monthly profit of \$60.

9. b. $\Delta L = 1.5$ and $\Delta P = (-.2 \times 1.5) = -.3$, a decrease in profit of \$300. The rate of change of profit is $-.3$ thousands of dollars/person, a negative rate of change.

9. c. $\Delta P = (-.3 \times 3) = -.9$ thousands of dollars, a decrease in profit of \$900 per month. If profit level is to remain unchanged, then $0 \approx -.9 - .12\Delta K$, so capital expenses must decrease by $\Delta K \approx -7.5$ thousands of dollars.

10. a.



$$\begin{aligned} h &= D \tan \theta \\ &= 100 \tan(57^\circ) \\ &= 154 \text{ ft} \end{aligned}$$

10. b. If $h = D \tan \theta$, the microscope equation for h is

$$\Delta h \approx \tan \theta \Delta D + D \sec^2 \theta \Delta \theta.$$

Since $\theta = 57^\circ$, $\tan \theta = 1.54$ and $\sec^2 \theta = 3.37$. Since $D = 100$, the microscope equation becomes

$$\Delta h \approx 1.54\Delta D + 337\Delta \theta.$$

This equation assumes θ is measured in radians. We assume D is certain only to $\Delta D = 1$ foot and θ is certain only to $\Delta\theta = 5^\circ = .087$ radians. Therefore h is certain only to

$$\Delta h \approx (1.54 \times 1) + (337 \times .087) = 30.86 \text{ feet.}$$

10. c. In the last calculation, the uncertainty ΔD contributed only 1.54 feet to the total Δh , while the uncertainty $\Delta\theta$ contributed more than 29.3 feet. It would therefore be more effective to improve the accuracy of the angle measurement θ .

Solutions: Chapter 4

Differential Equations

4.1 Modeling with Differential Equations

Single-species models

1. a. After 6 months there are 3644 rabbits.

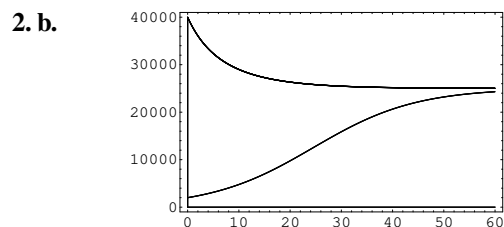
Δt	6 mos	24 mos
.1	3633.39	21785.11
.01	3643.15	22019.93
.001	3644.13	22043.71
.0001	3644.23	22046.09
.00001	3644.24	22046.33
.000001		22046.35

1. b. After 24 months there are 22 046 rabbits.

1. c. If $\Delta t = .001$, then using Euler's method $R(25.258) = 24998.63$ and $R(25.259) = 25001.13$, therefore it takes $t = 25.258$ months.

2. a.

Δt	6 mos	24 mos	60 mos
.1	3412.41	12197.19	24310.51
.01	3418.65	12231.55	24307.45
.001	3419.28	12234.99	24307.14
.0001	3419.34	12235.33	24307.11
.00001	3419.35	12235.37	



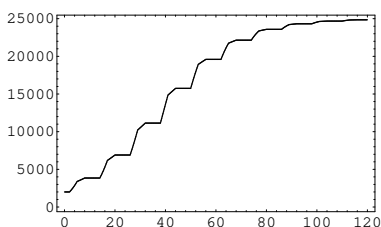
[Note: the above graphs were produced using a modification of SIRPLOT with $\Delta t = 2^{-8}$.]

3. a. This is a difficult question for a couple of reasons. First, students haven't been exposed to IF ... THEN expressions in programs in the text. Second, as time cycles through the different months, students must find a

way to test this. Modular arithmetic doesn't work because the time doesn't necessarily have an integer value.

The following is a QuickBasic program that plots R for 10 years ($n = 10$). The first eight lines set this value and also set the graphics parameters. The outer loop steps through the years; the inner loop steps through a single year with m steps per month. Notice that tests are made on the variable s , which is just t reduced to a value between 0 and 12 (and $s = 0$ represents January).

```
DEFDBL A-Z
SCREEN 9
COLOR 8, 7
n = 10          ' number of years
p = 12 * n      ' number of months
tinit = 0
tfin = p
WINDOW (tinit, 0)-(tfin, 30000)
h = 6
t = tinit
r = 2000
m = 2 ^ h      ' steps per month
q = p * m      ' total steps
deltat = (tfin - tinit) / q
FOR j = 1 TO n
    FOR i = 1 TO 12 * m
        s = t - 12 * (j - 1)
        IF s < 2 THEN rprime = 0
        IF s >= 2 AND s < 5 THEN
            rprime = .2 * r * (1 - r / 25000)
        IF s >= 5 AND s < 8 THEN
            rprime = .05 * r * (1 - r / 25000)
        IF s >= 8 AND s < 12 THEN
            rprime = 0
        deltar = rprime * deltat
        LINE (t, r)-(t + deltat, r + deltar)
        t = t + deltat
        r = r + deltar
    NEXT i
NEXT j
```



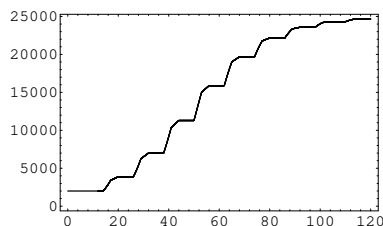
During each year, there is a period of no growth, a period of relatively steep growth, and a period of slower growth.

The figure above was produced by Mathematica, using a translation of the QuickBasic code. Here is the Mathematica code:

```
n = 10;
p = 12 * n;
tinit = 0;
tfin = p;
h = 6;
t = tinit;
r = 2000;
m = 2 ^ h;
q = p * m;
deltat = (tfin - tinit) / q;
Curve = {};
For[j = 1, j <= n, j++,
  For[i = 1, i <= 12 * m, i++,
    s = t - 12 * (j - 1);
    If[s < 2, rprime = 0];
    If[2 <= s < 5,
      rprime = .2 * r * (1 - r/25000)];
    If[5 <= s < 8,
      rprime = .05 * r * (1 - r/25000)];
    If[8 <= s < 12, rp = 0];
    deltar = rprime * deltat;
    Curve = Append[
      Curve, Line[{
        {t, r},
        {t + deltat, r + deltar}
      }
    ];
    t = t + dt;
    r = r + dr
  ]
];
Show[
  Graphics[{Curve}],
  Frame -> True
]
```

3. b. There are several ways to deal with the first year under the assumption that the rabbits won't reproduce then. One way is to change the outer loop so that it starts with year 2 and insert two new lines that increment t from 0 to 12 and draw a horizontal line for that time interval. Specifically, replace

```
FOR j = 1 TO n
by
  LINE (t, r)-(t + 12, r)
  t = t + 12
  FOR j = 2 TO n
```



4. a. $P' = .02P$.

4. b. $P'(1990) = .02 \times 5 = .1$ billion people per year.

4. c. Δt	1980	2000	2040	2230
.1	4.0928	6.1058	13.5778	604.6466
.01	4.0936	6.1069	13.5900	607.2605
.001				607.5229
.0001				607.5491

One decimal place accuracy in billions for the years 1980, 2000 and 2040 was obtained with $\Delta t = .1$, for the year 2230, with $\Delta t = .001$.

5. a. $P'(1990) = .015(5)^{1.2} \approx .1035$ billion people per year (or 103.5 million people per year).

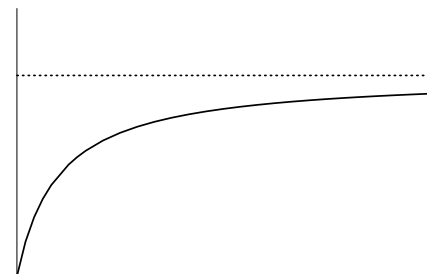
5. b. Δt	1980	2000	2040
.1	4.0813	6.1760	15.9113
.01	4.0821	6.1766	15.9373

One decimal place accuracy in billions for the years 1980, 2000 and 2040 was obtained with $\Delta t = .1$.

5. c. If $\Delta t = .1$, then $P(2230) \approx 1.979 \times 10^{10}$. If $\Delta t = .01$, then $P(2230) \approx 2.573 \times 10^{11}$. The second estimate is more than 10 times larger than the first.

Two-Species models

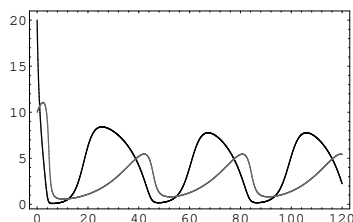
6. WARNING: QuickBasic has OVERFLOW problems. Near the bottom of one of the curves—where the curve turns rapidly—a large deltat allows x or y to become negative, at which point it quickly goes to $-\infty$.



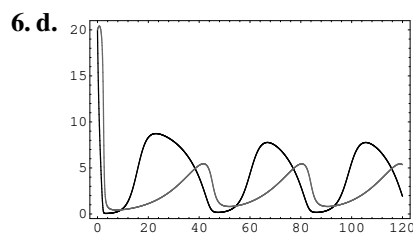
6. a. Let $D(x) = cx/(x+d)$; its graph is shown above. Then, when x is small, $x+d \approx d$ so $D(x) \approx cx/d$. This is a linear function of x with slope c/d . When x is large, $x/(x+d) \approx 1$ so $D(x) \approx c$. This function fits the criteria given for $D(x)$.

6. b. In the absence of foxes, $y = 0$ and the growth of rabbits is governed by $x' = ax(1 - x/b)$, a logistic equation. Next, the term $D(x) \cdot y$ describes the loss of rabbits due to predation. Finally, the fox equation $y' = ey(1 - y/fx)$ is also a logistic equation—in particular, the carrying capacity is the factor fx , which is proportional to the number of rabbits, x .

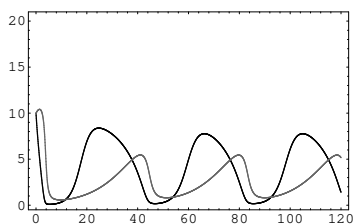
6. c. Since x is the number of rabbits *divided* by 100, $x(0) = 20$. Over time the rabbit and fox populations *become* periodic.



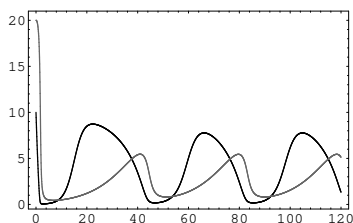
Note: $x(t)$ is the darker graph.



$x(0) = 20$,
 $y(0) = 20$



$x(0) = 10$,
 $y(0) = 10$

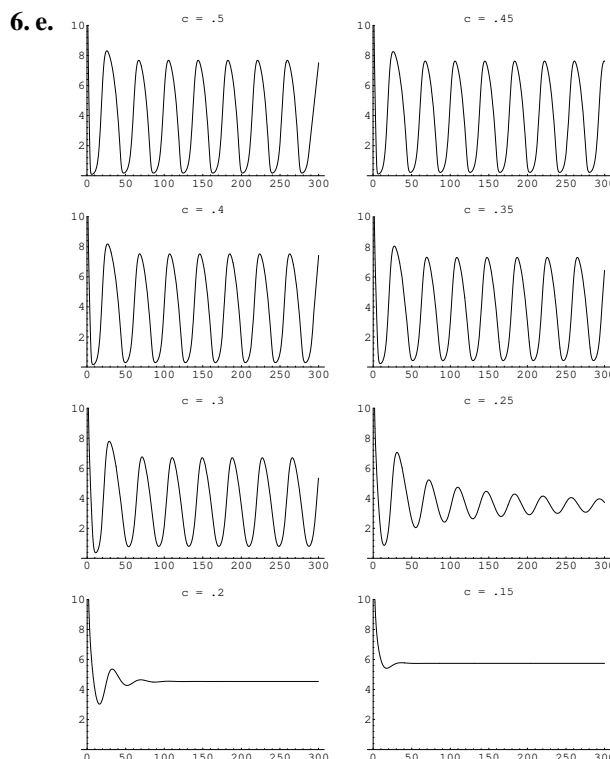


$x(0) = 10$,
 $y(0) = 20$

[Note: these graphs were produced using $\Delta t = 2^{-4}$.]

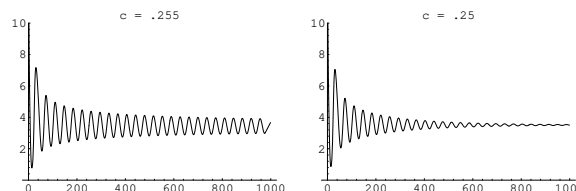
The eventual long-term behavior does not depend upon the initial conditions. The populations governed by May model equations remain periodic in the long-term, while the populations governed by the modified Lotka-Volterra equations that appear in the text approach a constant

value, that is, in the long run the periodic behavior of the populations is dampened until it no longer exists.



The graphs above plot $x(t)$ for the indicated values of c on the same vertical and horizontal scales. As c decreases, $x(t)$ switches from periodic to a decaying oscillation (the “bifurcation”). This bifurcation occurs somewhere between $c = .3$ and $c = .25$. By the time $c = .2$, the oscillations are decaying very quickly.

We get more precise information about the bifurcation point by sampling near $c = .25$ using a longer time scale. The figures below show that the bifurcation occurs between $c = .250$ and $c = .255$.

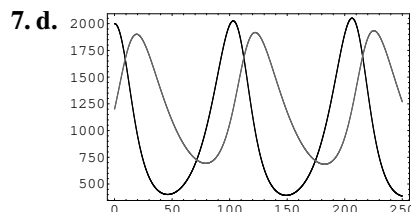


7. a. If we start with the Lotka–Volterra equations in the text, we need explain only the differences introduced by the assumptions made in this question. The new assumption is simply that the hares experience constant per capita growth in the absence of lynxes. But when $L = 0$, H' reduces to $H' = aH$, which is indeed constant per capita growth. (By contrast, in the (modified) Lotka–Volterra equations in the text, H' reduced to a logistic differential equation.)

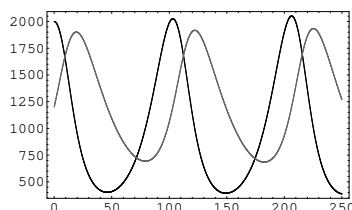
7. b. Since $H' = aH + \dots$, the multiplier a must convert hares to hares per month. Thus a must be measured in hares per month per hare. Since $H' = \dots + bHL$, b must convert hare-lynxes to hares per month. Thus b must be measured in hares per month per hare-lynx.

Since $L' = cHL + \dots$, c must convert hare-lynxes to lynxes per month. Thus, c must be measured in units of lynxes per month per hare-lynx. Since $L' = \dots - dL$, d must be measured in units of lynxes per month per lynx.

7. c. The solution graph is in the text.



Lynxes are in gray.



Scale is shown for hares; lynx scale is 1/60th of this.

The populations return periodically to their original values. The maximum and minimum number of hares and lynxes vary with the initial conditions.

[Note: $\Delta t = .1$ and L is 60 times the actual number of lynxes in these graphs.]

7. e. Using a DO WHILE $H_{\text{prime}} > 0$ loop, we find the maximum number of hares is 2830.

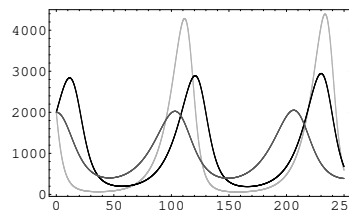
Δt	t	Maximum H
.1	11.6	2839.33
.01	11.5	2830.93
.001	11.495	2830.09
.0001	11.4938	2830.01

The length of *one* period for the hare population is ≈ 108.7 months. The maximum number of lynxes is 40.

Δt	t	Maximum L
.1	28.1	40.60
.01	28.09	40.38
.001	28.088	40.36

The hare and lynx populations have the same period of ≈ 108.7 months.

7. f.

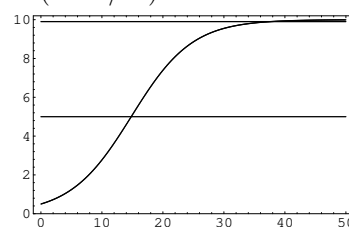


The hare populations are periodic. The period varies with the initial size of the lynx population. If $L(0) = 10$, then the period, $T \approx 108.7$ months. If $L(0) = 20$, then $T \approx 102.95$ months. If $L(0) = 50$, then $T \approx 122.2$ months.

Fermentation

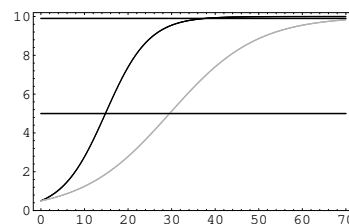
8. a. $Y' = .2Y(1 - Y/10)$.

8. b.



$Y(t)$ is within 1% of the carrying capacity when $Y(t) \geq 9.9$ lbs. $Y(t) \approx 4.999751$ when $t = 14.73$; $Y(t) \approx 9.900001$ when $t = 37.69$.

8. c.



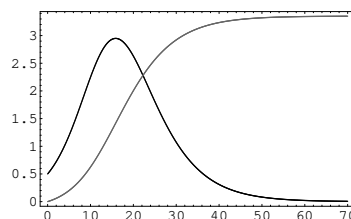
Here $Y' = .1Y(1 - Y/10)$. We find $Y(t) \approx 4.9993$ when $t = 29.45$; $Y(t) \approx 9.900022$ when $t = 75.39$. When the natural growth rate is cut in half, the yeast takes about twice as long to reach a given level.

9. a. $A' = .05Y$.

9. b. If there are Y lbs of yeast and A lbs of alcohol, then $.1AY$ lbs of yeast will die in an hour; therefore

$$Y' = .2Y(1 - Y/10) - .1AY.$$

9. c.



$\Delta t = .1$; $Y(t)$ is the darker graph.

Yeast reaches a maximum of ≈ 2.93 lbs in ≈ 15.8 hours.

Δt	t	Maximum Y
.1	15.9	2.9489
.01	15.80	2.9351
.001	15.792	2.9337

Fermentation ends when no more alcohol is being produced. Since $A' = .05Y$, $A' = 0$ if $Y = 0$, and fermentation ends when $Y \approx 0$. Therefore, any answer will depend on the *specified* criterion, $Y \leq n$, for some n , a real number, which is *flexible*.

If we consider fermentation to be over when $Y \leq .01$ lbs, then fermentation ends at $t \approx 65.9$ hrs with $A \approx 3.342$ lbs.

Δt	t	$A(t)$
.1	65.6	3.3482
.01	65.88	3.3429
.001	65.911	3.3424

If we consider fermentation to be over when $Y \leq .0001$ lbs, then fermentation ends at $t \approx 100$ hrs with $A \approx 3.346$ lbs.

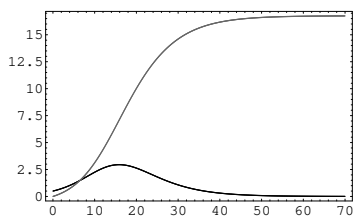
Δt	t	$A(t)$
.1	99.3	3.3518
.01	100.05	3.3466
.001	100.127	3.3461

10. Changing the rates of toxicity and alcohol production, we get the differential equations

$$Y' = .2Y(1 - Y/10) - .02AY$$

$$A' = .25Y$$

If we use the criterion that fermentation has ended when $Y \leq .01$ lbs, then fermentation ends after $t \approx 65.9$ hours, with $A \approx 16.71$ lbs. The fermentation time remains the same while the amount of alcohol produced increases.



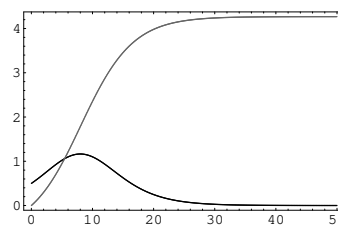
Δt	t	$Y(t)$	$A(t)$
.1	65.6	9.9066×10^{-3}	16.7409
.01	65.88	9.9943×10^{-3}	16.7146
.001	65.911	9.9989×10^{-3}	16.7120

If only the alcohol production rate is changed, then the differential equations are

$$Y' = .2Y(1 - Y/10) - .1AY$$

$$A' = .25Y.$$

Fermentation is over after $t \approx 34.7$ hours with $A \approx 4.24$ lbs. The length of fermentation and amount of alcohol produced are both less than in the original problem.



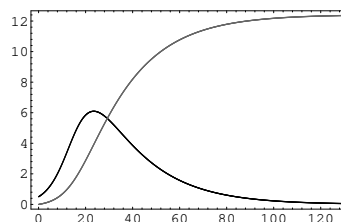
Δt	t	$Y(t)$	$A(t)$
.1	34.5	9.8708×10^{-3}	4.2583
.01	34.74	9.9846×10^{-3}	4.2433
.001	34.763	9.9980×10^{-3}	4.2419

If only the toxicity is changed, then the differential equations are

$$Y' = .2Y(1 - Y/10) - .02AY$$

$$A' = .05Y.$$

Fermentation is over after $t \approx 165.2$ hours with $A \approx 12.41$ lbs. The length of fermentation and amount of alcohol produced are both larger than in the original problem.



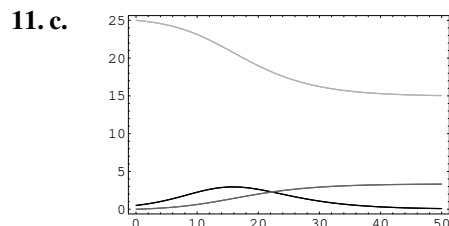
Δt	t	$Y(t)$	$A(t)$
.1	164.9	9.9919×10^{-3}	12.4077
.01	165.21	9.9958×10^{-3}	12.4062
.001	165.234	9.9995×10^{-3}	12.4061

[Note: $\Delta t = .1$ for each set of graphs.]

11. a. $S' = -.15Y$.

11. b. Since S lbs of sugar are needed to “carry” $.4S$ lbs of yeast, we see $.4S = 10$ so $S = 25$. That is, 25 lbs of sugar are needed to maintain a carrying capacity of 10 lbs of yeast. Likewise, 2.5 lbs of sugar are needed to maintain a carrying capacity of 1 lb of yeast. The yeast differential equation therefore becomes

$$Y' = .2Y \left(1 - \frac{Y}{.4S} \right) - .1AY.$$



Most of the sugar is unconsumed, but all the yeast disappears. When $t = 68.5$ hrs, $Y \approx .01$ lbs, $S \approx 15.36$ lbs, and $A \approx 3.21$ lbs.

Δt	t	$Y(t)$	$A(t)$	$S(t)$
.1	68.2	9.8992×10^{-3}	3.2169	15.3494
.01	68.47	9.9963×10^{-3}	3.2126	15.3622
.001	68.502	9.9998×10^{-3}	3.2122	15.3635

Newton's Law of Cooling

12. a. We know $Q' = \pm k(Q - 20)$ at the start. When $Q > 20$ (so $k(Q - 20)$ is positive), Q' must be negative (because Q is decreasing). This means we must choose the minus sign in the differential equation.

12. b. After 20 minutes the temperature of the coffee is $\approx 29.47^\circ\text{C}$.

Δt	$T(20)$
.1	29.4733°C
.01	29.4640°C
.001	29.4735°C
.0001	29.4734°C

12. c. It takes about 19.46 minutes.

Δt	t	$T(t)$
.1	19.4	29.96152°C
.01	19.45	29.99937°C
.001	19.459	29.99913°C
.0001	19.4591	29.99990°C

13. $Q(5) \approx 70.13$; $Q(10) \approx 82.69$.

Δt	$Q(5)$	$Q(10)$
.1	70.3348°F	82.8385°F
.01	70.1544°F	82.7065°F
.001	70.1365°F	82.6934°F
.0001	70.1347°F	82.6920°F

14. a. In this example, $Q - A$ is positive. Since Q will be *decreasing*, the differential equation for Q needs a minus sign. By contrast, A will be *increasing*, so the differential equation for A needs a plus sign.

14. b.

Δt	t	$Q(t)$	$A(t)$
.1	7.5	39.9575°C	36.6808°C
.01	7.6	39.9961°C	36.6680°C
.001	7.61	39.9998°C	36.6667°C

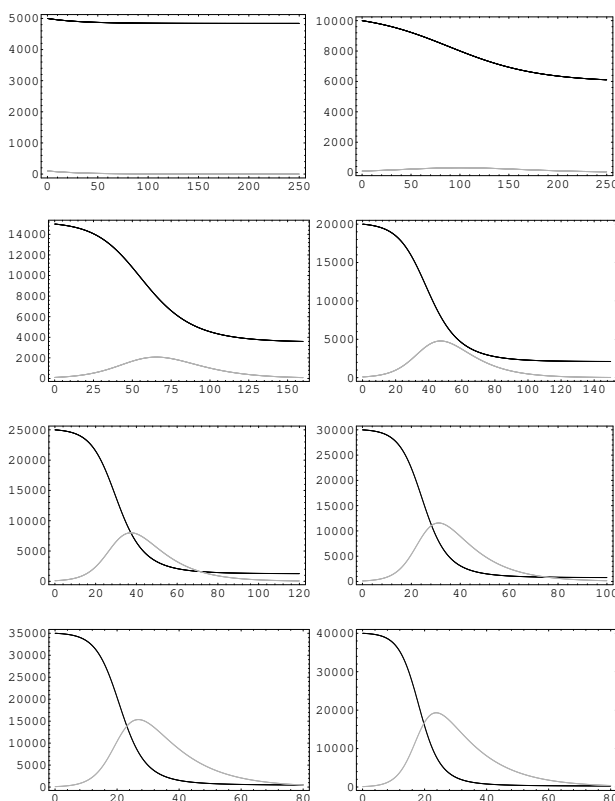
14. c. The temperature of the coffee—and the air—eventually become 37.5°C , and this happens after about 25 minutes. This was found using a DO While loop testing $Q' < -.001$.

S-I-R Revisited

15. There are 105 susceptibles after 40 days. The largest number of people are infected when $t = 5.3$ days. When the largest number of people are infected, $0 = I' = aSI - bI = I(aS - b)$ so either $I = 0$ or $S = b/a$. I cannot be zero so $S = b/a = .00001/.08 = 8000$. Thus, there are 8000 susceptibles, and we've determined this without using Euler's method.

16. As $t \rightarrow \infty$, $I(t) \rightarrow 0$ and $S(t) \rightarrow 0$.

17. In the following figures, the value of S increases as you read from left to right and then down the page.



In these graphs, it is clear that as $S(0)$ increases, the amount of time it takes for $S(t)$ to level off decreases. [Note: $\Delta t = .1$ in the above graphs.]

18. $I(t)$ is the gray graph, graphed along with $S(t)$ in the previous solution.

$S(0)$	I_{max}	P_{max}	t_{max}
5000	100	.02	0
10000	315	.03	100.2
15000	2071	.14	65.3
20000	4770	.24	47
25000	7985	.32	37.2
30000	11526	.38	31
35000	15293	.44	26.7
40000	19225	.48	23.5
45000	23282	.52	21.1

I_{max} is the whole number nearest to the approximation $I(t)$ after the first decimal place has stabilized using successively smaller Δt . t_{max} is the time I_{max} occurred rounded to the nearest tenth.

Constructing Models

19. Yes. Let $s(t)$ be the amount of money at time t . We know $s(0)$ and we need to know the interest rate i . If we know i , then $s'(t) = i \cdot s(t)$. (Note: i may be fixed or it may vary with t . In either case we can solve the differential equation by Euler's method.)

20. Yes. Let $d(t)$ be the diameter of the spot at time t . We know $d(0) = .25$ inch; we need to know the rate at which $d(t)$ is growing. If $d'(t) = f(t)$ for some known function of t , we then have an initial value problem that can be solved by Euler's method.

21. Yes. We need to know the rate of change of the fecal bacteria, $f'(t)$, and the rate of change of coliform concentrations, $c'(t)$, and the initial values of each, $f(0)$ and $c(0)$. (In practice, these rates might be estimated as constants from samples of the values of f and c over some time period.)

22. No. Temperature and rainfall—and their rates of change—depend on too many factors. We would need to know—at a minimum—how temperature is changing from place to place, as well as from time to time. In other words, what we need are *partial* differential equations.

23. No. The lottery number is random. Likewise, the “rate of change” of the lottery number from day to day should be random, too.

24. Yes. Many students may not be familiar with the physical/mathematical background, but this should still be within their reach. Each object (sun, moon, earth) has three position and three velocity coordinates; the total is 18. We get nine differential equations by setting

the derivative of each position variable equal to the corresponding velocity. The remaining nine differential equations will then describe the rate of change of each velocity variable in terms of all the position and velocity variables. The last sentence of the problem declares that this can be done—though it doesn't indicate how, of course. (In fact, Newton's laws of motion and gravitation provided the details.) Finally, if the 18 position and velocity variables are given at one instant, we have a legitimate initial value problem.

4.2 Solutions of Differential Equations

1. a. $y = (\frac{1}{5} - y)^{-1}$, so $y' = -(\frac{1}{5} - t)^{-2} \cdot (-1) = (\frac{1}{5} - t)^{-2} = y^2$. Also, $y(0) = (\frac{1}{5} - 0)^{-1} = 5$.

1. b. $y' = (\frac{1}{25} - 2t)^{-3/2} = y^3$, and $y(0) = (\frac{1}{25} - 0)^{-1/2} = 5$.

1. c. $y' = (\frac{1}{125} - 3t)^{-4/3} = y^4$, and $y(0) = (\frac{1}{125} - 0)^{-1/3} = 5$.

1. d. $y(t) = \left(\frac{1}{5(n-1)} - (n-1)t\right)^{-1/(n-1)}$.

1. e. $y(t) = \left(\frac{1}{c(n-1)} - (n-1)t\right)^{-1/(n-1)}$.

2. a. $y' = t^2$ and $y(0) = 0 + 5 = 5$.

2. b. $y' = t^3$ and $y(0) = 0 + 5 = 5$.

2. c. $y' = t^4$ and $y(0) = 0 + 5 = 5$.

2. d. $y(t) = \frac{1}{n+1}t^{n+1} + 5$.

2. e. $y(t) = \frac{1}{n+1}t^{n+1} + C$.

3. a. $x' = -\sin t = -y$, $y' = \cos t = x$, $x(0) = \cos(0) = 1$, $y(0) = \sin(0) = 0$.

3. b. $x' = -\sin(t + \pi/2) = -y$, $y' = \cos(t + \pi/2) = x$, $x(0) = \cos(\pi/2) = 0$, $y(0) = \cos(\pi/2) = 1$.

4. In this question, we are assuming that the derivative of 10^x is $2.3 \cdot 10^x$.

4. a. $y' = 5(2.3 \cdot 10^y) = 2.3y$, $y(0) = 5 \cdot 10^0 = 5$.

4. b. $y' = C(2.3 \cdot 10^y) = 2.3y$, $y(0) = C \cdot 10^0 = C$.

4. c. $y' = 5(2.3 \cdot 10^{-t} \cdot (-1)) = -2.3y$, $y(0) = 5 \cdot 10^0 = 5$.

4. d. $y' = 5(2.3 \cdot 10^{t^2} \cdot 2t) = 4.6ty$, $y(0) = 5 \cdot 10^0 = 5$.

5. a. We need $y(3) = \sqrt{3+C} = 17$. Thus $3+C = 17^2 = 289$ so $C = 286$.

5. b. $C = -1/5$.

5. c. $C = -7/3$.

World Population Growth with Parameters

6. a. If $P(t) = (1.0202)^t$, then $P'(0) \approx .019998687$.

h	$P'(0)$
.1	.01999870
.01	.01999869
.001	.01999869

6. b. If $P(t) = (1.0202013)^t$, then $P'(0) \approx .019999961$.

h	$P'(0)$
.1	.019999974
.01	.019999961
.001	.019999961

7. a. $dy/du = .69 \cdot 2^u$, $du/dt = 1/.69$, so $dy/dt = .69 \cdot 2^{t/.69} (1/.69) = 2^{t/.69} = y$.

7. b. Let $u = kt/.69$; then $y = 2^u$. $dy/du = .69 \cdot 2^u$ and $du/dt = k/.69$, so $dy/dt = .69 \cdot 2^{kt/.69} (k/.69) = k2^{kt/.69} = ky$.

7. c. Let $u = kt/.69$; then $P = A \cdot 2^u$. $dP/du = .69A \cdot 2^u$ and $du/dt = k/.69$, so

$$\frac{dP}{dt} = .69A \cdot 2^{kt/.69} \cdot \frac{k}{.69} = kA \cdot 2^{kt/.69} = kP.$$

Furthermore, $P(0) = A \cdot 2^0 = A$.

8. a. $dy/du = 2.3 \cdot 10^u$ and $du/dt = 1/2.3$, so $dy/dt = 2.3 \cdot 10^{t/2.3} (1/2.3) = 10^{t/2.3} = y$.

8. b. $dy/du = 2.3 \cdot 10^u$ and $du/dt = k/2.3$, so $dy/dt = 2.3 \cdot 10^{t/2.3} (k/2.3) = k10^{t/2.3} = ky$.

8. c. Let $u = kt/2.3$; then $P = A \cdot 10^u$. $dP/du = 2.3(A \cdot 10^u)$, $du/dt = k/2.3$, so

$$\frac{dP}{dt} = 2.3 \left(A \cdot 10^{kt/2.3} \right) \cdot \frac{k}{2.3} = kA \cdot 10^{kt/2.3} = kP$$

and $P(0) = A \cdot 10^0 = A$.

9. a. Solve for t in $10 = P(t) = 5 \cdot 2^{kt/.69}$:

$$10 = 5 \cdot 2^{kt/.69} \text{ so } 2 = 2^{kt/.69};$$

$$\text{hence } 1 = kt/.69 \text{ and } t = .69/k$$

9. b. The formula for P is now $P(t) = A \cdot 2^{kt/.69}$, and we want to solve for t in $2A = P(t) = A \cdot 2^{kt/.69}$:

$$2A = A \cdot 2^{kt/.69} \text{ so } 2 = 2^{kt/.69}.$$

We have arrived at the same point we did in the previous part; therefore, the doubling time t is the same (and is thus independent of the initial population size A): $t = .69/k$.

9. c. The relation between the doubling time t and k is $t(k) = .69/k$. The error propagation equation is therefore

$$\Delta t \approx t'(k) \Delta k \approx -.69/(\cdot 02)^2 \Delta k = -1725 \Delta k.$$

We have used the fact that $t'(k) = -.69/k^2$. Since the units for Δk are 1/yr, the units for the coefficient 1725 are yr^2 .

Perhaps the *relative* error is a better indicator:

$$\frac{\Delta t}{t} \approx \frac{-.69/k^2}{.69/k} \Delta k = -\frac{\Delta k}{k}.$$

Thus the percentage errors are equal and of opposite sign.

Newton's Law of Cooling

10. $dy/du = .69 \cdot 2^u$ and $du/dt = -.1/.69$, so

$$\frac{dy}{dt} = .69 \cdot 2^{-.1t/.69} \cdot \frac{-.1}{.69} = -.1 \cdot 2^{-.1t/.69} = -.1y.$$

11. a. Let $u = -.1t/.69$; then $T = 70 \cdot 2^u + 20$ and $T - 20 = 70 \cdot 2^u$. Therefore $dT/du = 70(.69 \cdot 2^u)$ and $du/dt = -.1/.69$, so

$$\begin{aligned} \frac{dT}{dt} &= 70(.69 \cdot 2^{-.1t/.69}) \cdot \frac{-.1}{.69} \\ &= -.1(70 \cdot 2^{-.1t/.69}) = -.1(T - 20) \end{aligned}$$

and $T(0) = 70 \cdot 2^0 + 20 = 90$.

11. b. $T(20) = 70 \cdot 2^{-.1 \times 20/.69} + 20 \approx 29.39^\circ\text{C}$. The result found in the earlier exercise (12b in section 4.1) was 29.47°C ; these are different.

11. c. We must solve for t in the equation

$$30 = T(t) = 70 \cdot 2^{-.1t/.69} + 20 \text{ so } 1/7 = 2^{-.1t/.69};$$

$$\text{hence } 2^{-2.8} = 2^{-.1t/.69} \text{ and } -2.8 = -.1t/.69$$

Thus $t = 19.32$. The result found in problem 12c was 19.46 minutes.

12. a. If $u = -.2t/.69$, then $Q = 90 - 54 \cdot 2^u$. Therefore $dQ/du = 54(.69 \cdot 2^u)$, $du/dt = -.2/.69$, and

$$\frac{dQ}{dt} = -.2(54 \cdot 2^{-.2t/.69}) = -.2(Q - 90).$$

Furthermore, $Q(0) = 90 - 54 \cdot 2^0 = 36^\circ\text{F}$.

12. b. $Q(5) \approx 70.2249^\circ\text{F}$ and $Q(10) \approx 82.7583^\circ\text{F}$. The values from exercise 13 were $Q(5) \approx 70.13^\circ\text{F}$ and $Q(10) \approx 82.69^\circ\text{F}$.

13. $Q(t) = (B - A)2^{-kt/.69} + A$.

A Leaking Tank

14. a. The “leakage rate” is just the rate V' at which the volume is changing. It is stated that this is proportional to the square root of the height h . But $V = \pi r^2 h$ and r is fixed, so h is proportional to V . Therefore, the leakage rate must be proportional to the square root of V :

$$V'(t) = -k\sqrt{V(t)}.$$

To indicate V is decreasing, we take k to be positive and use a minus sign in the equation.

14. b. Insert the known values of $V'(0)$ and $V(0)$ into the differential equation to determine the value of k . It is given that $V'(0) = -2$ (cubic feet per hour). To determine $V(0)$, note that $h = 5$ feet (and $r = 1.5$ feet) then, so

$$V(0) = \pi r^2 h = \pi(1.5)^2 \times 5 \approx 35.342917$$

Therefore

$$k = \frac{-V'(0)}{\sqrt{V(0)}} = \frac{2}{5.9449909} = .33641766 \dots$$

or $k \approx .3364$.

15. a.	Δt	$V(12)$	$V(24)$
	.1	15.3907	3.6046
	.01	15.4154	3.6375
	.001	15.4178	3.6408
	.0001	15.4180	3.6412

$V(12) \approx 15.42$ so $V(0) - V(12) \approx 19.92$ cubic feet have leaked out in 12 hours. $V(24) \approx 3.64$ so $V(0) - V(24) \approx 31.70$ cubic feet have leaked out in 24 hours.

15. b. If we consider the cylinder to be empty when the volume is $V < .0001$, then it takes ≈ 35.29 hours for the tank to empty. The following is a table of results from a DO-WHILE loop with various time steps Δt .

Δt	t
.1	35
.01	35.26
.001	35.283
.0001	35.2851

16. a. If $0 \leq t \leq C$, then

$$V'(t) = -\frac{k^2}{2}(C - t) = -k\left(\frac{k}{2}(C - t)\right) = -k\sqrt{V(t)}.$$

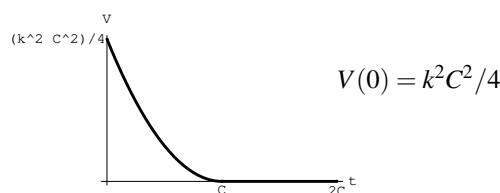
If $C < t$, then $V'(t) = 0 = -k\sqrt{V(t)}$.

16. b. We can determine the value of C by setting $t = 0$ in the formula $V'(t) = -k^2(C - t)/2$:

$$-2 = \frac{-k^2 C}{2} \text{ so } C = 4/k^2 = 35.34291 \dots$$

According to the formula for $V(t)$, C is the time it takes for the tank to empty. Thus, it takes about 35.34 hours. In part b of the previous question, we determined that it took about 35.29 hours for the tank's volume to fall below .0001 cubic feet. Thus, the answers were not identical, but these results are consistent.

16. c.



Motion

17. $v'(t) = -g$ and $v(0) = 0 + v_0 = v_0$.

18. $x(t) = -\frac{g}{2}t^2 + v_0 t$.

19. $x(t) = -\frac{g}{2}t^2 + v_0 t + x_0$

20. a. $x_0 = 200$ meters and $v_0 = 0$ meters per second. Therefore,

$$x(t) = -\frac{g}{2}t^2 + 200 = -4.9t^2 + 200 \text{ meters.}$$

20. b. $x(1) = 195.1$ m, so the body has fallen 4.9 meters. $x(2) = 180.4$ m, so the body has fallen 19.6 meters.

20. c. Solving the equation $x(t) = 0$ for t , we find it takes about 6.39 seconds for the body to reach the ground.

21. One way to do this is to show that dv/dt and $-g - bv$ are identical, as functions of t . First,

$$\frac{dv}{dt} = \frac{g}{b} \cdot .69 \cdot 2^{-bt/.69} \cdot \frac{-b}{.69} = -g \cdot 2^{-bt/.69}.$$

Second,

$$-g - bv = -g - g \left(2^{-bt/.69} - 1 \right) = -g \cdot 2^{-bt/.69}.$$

Finally, the initial condition:

$$v(0) = \frac{g}{b} \left(2^{-b \cdot 0 / .69} - 1 \right) = (2^0 - 1) = 0,$$

as required.

22. a. Note: the student need not *derive* the formula for $x(t)$ that is given in the problem. It is enough simply to check that the given formula satisfies the initial condition $x(0) = x_0$ and that its derivative is the function $v(t)$ given in the previous problem. First,

$$x(0) = x_0 - \frac{g}{b} \cdot 0 - \frac{g}{b^2} (2^0 - 1) = x_0 - 0 - 0 = x_0.$$

Next,

$$\begin{aligned} x'(t) &= -\frac{g}{b} - \frac{g}{b^2} \left(.69 \cdot 2^{-bt/.69} \cdot \frac{-b}{.69} \right) \\ &= -\frac{g}{b} - \frac{g}{b^2} 2^{-bt/.69} = v(t). \end{aligned}$$

22. b. $x(1) \approx 195.59$ m., so the body has fallen ≈ 4.41 meters in the first second. The earlier estimate was 4.9 m. $x(2) = 183.07$, so the body has fallen 16.93 meters in the first 2 seconds. The earlier estimate was 19.6 meters.

22. c. Solving $x(t) = 0$ for t , we find it takes about 8.0993, or 8.1, seconds for the body to reach the ground. Air resistance adds about 1.6 seconds.

23. a. When the velocity doesn't change, we have $v' = 0$. But $v' = -g - bv$, so $v' = 0$ implies $-g - bv_T = 0$. Hence $v_T = -g/b$.

23. b. $v_T = -9.8/.2 = -49$ meters per second. (This is about 120 miles per hour.)

24. a. $x'(t) = -ac \sin(ct) = v(t)$,
 $v'(t) = -ac^2 \cos(ct) = -c^2 x(t)$,
 $x(0) = a \cos(0) = a$, $v(0) = -ac \sin(0) = 0$.

24. b. $-a \leq x \leq a$.

25. a. Larger values of c shorten the period; smaller values lengthen it.

25. b. When $c = 1$ the period is 2π .

25. c. When $c = 2$ the period is π ; when $c = 3$ the period is $2\pi/3$. The period does not depend on a .

25. d. $T = 2\pi/c$, where T is the period.

26. a. $T = 2\pi/\sqrt{k/m} = 2\pi\sqrt{m}/\sqrt{k}$.

26. b. Doubling m *increases* the period of the motion. Intuitively, a heavier mass should take longer to complete one cycle.

26. c. Doubling k *decreases* the period of the motion. Intuitively, a stiffer spring (i.e., one with a higher k) will oscillate faster—and thus have a shorter period.

26. d. We need the relative error propagation equation for T in terms of m . We know

$$T = 2\pi\sqrt{k}m^{-1/2}, \text{ so } \frac{\partial T}{\partial m} = -\pi\sqrt{k}m^{-3/2}.$$

Therefore

$$\frac{\Delta T}{T} \approx \frac{-\pi\sqrt{k}m^{-3/2}}{2\pi\sqrt{k}m^{-1/2}} \Delta m = -\frac{1}{2} \frac{\Delta m}{m}.$$

Accordingly, if we know the mass m only to within 5%, we can calculate the actual value of the period T only to within 2.5%.

4.3 The Exponential Function

The Exponential Functions b^t

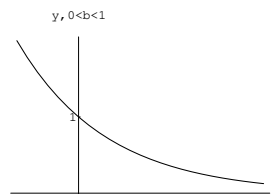
1. a. Estimating $f'(0)$ by $(f(0+h) - f(0-h))/2h$ with a sequence of h values that decrease toward 0, we find $f'(0) \approx .996948 \dots < 1$.

1. b. Here $f'(0) \approx 1.000631 \dots > 1$.

1. c. According to the argument in the text on page 229, if $a < b$, then the slope at the origin of a^t is less than the slope of b^t there. Since the slope at the origin of e^t is exactly 1, the slope of $(2.71)^t$ is less than one, and the slope of $(2.72)^t$ is more than one, it follows that $2.71 < e < 2.72$.

2. a. When $f(t) = .5^t$, $f'(0) = k_{.5} = -.693 \dots$. Likewise, $k_{.75} = -.287 \dots$ and $k_{.9} = -.105 \dots$.

2. b.



Suppose $0 < b < 1$. Then as t goes from $-\infty$ to $+\infty$, $y = b^t$ is strictly decreasing, so $y' < 0$. But $y' = k_b b^t$; since $b^t > 0$, k_b must be negative for $0 < b < 1$.

Differentiating Exponential Functions

3. a. $21e^{3x}$.

3. b. Cke^{kx} .

3. c. $1.5e^t$.

3. d. $3e^{2t}$.

3. e. $6e^{3x} - 6e^{2x} = 6e^{2x}(e^x - 1)$.

3. f. $-\sin(t) \cdot e^{\cos(t)}$.

4. a. $f_x = ye^{xy}, f_y = xe^{xy}$.

4. b. $f_x = 6xe^{2y}, f_y = 6x^2e^{2y}$.

4. c. $f_u = e^u \sin(v), f_v = e^u \cos(v)$.

4. d. $f_u = \sin(v) \cdot e^{u \sin(v)}, f_v = u \cos(v) \cdot e^{u \sin(v)}$.

Powers of e

5. a. $\exp(2x+3) = e^3 \cdot e^{2x}$ by Theorems 1 and 2.

5. b. $(\exp(x))^2 = e^{2x}$ by Theorems 1 and 2.

5. c. $\exp(17x)/\exp(5x) = e^{12x}$ by Theorems 2 and 3.

6. Let $s = -t$; then $\exp(t) = \exp(-s)$. As $t \rightarrow -\infty, s = -t \rightarrow +\infty$ and

$$\lim_{t \rightarrow -\infty} \exp(t) = \lim_{s \rightarrow +\infty} \exp(-s).$$

Now, by Theorem 3, $\exp(-s) = 1/\exp(s)$; by Theorem 2, $1/\exp(s) = 1/e^s$. Thus

$$\lim_{t \rightarrow -\infty} \exp(t) = \lim_{s \rightarrow +\infty} \exp(-s) = \lim_{s \rightarrow +\infty} 1/e^s = 0.$$

7. a. $P'(t) = r \cdot \exp(rt) = r \cdot P$
 $P(0) = \exp(0) = 1$.

7. b. $dQ/du = ru^{r-1}$ and $du/dt = \exp(t)$. Therefore,

$$dQ/dt = r(\exp(t))^{r-1} \cdot \exp(t) = r(\exp(t))^r = r \cdot Q.$$

Furthermore, $Q(0) = (\exp(0))^r = 1$.

7. c. Since $P(t) = Q(t)$ for every t , we can set $t = s$. Then

$$\exp(rs) = P(s) = Q(s) = (\exp(s))^r,$$

as required.

Solving $y' = ky$ Using e^{kt}

8. a. $P' = .009P, P(0) = 37.5$,
 $A' = .0216A, A(0) = 15$.

8. b. $P(t) = 37.5e^{.009t}, A(t) = 15e^{.0216t}$.

8. c. Since the year 1985 corresponds to $t = 0$ in the above equations, we find the populations in 2005 by evaluating $P(20)$ and $A(20)$ and those in 1965 by evaluating $P(-20)$

and $A(-20)$:

$$P(20) \approx 44.8957 \dots \text{million people,}$$

$$A(20) \approx 23.1050 \text{ million people,}$$

$$P(-20) \approx 31.3226 \text{ million people,}$$

$$A(-20) \approx 9.7381 \text{ million people.}$$

9. a. $P' = .0247P$ gm/hr, $P(0) = 32$ gm.

9. b. $P(t) = 32e^{.0247t}$ gm.

9. c. $P(.5) \approx 32.40$ gm, $P(2) \approx 33.62$ gm.

10. a. $R' = -(1/2337)R$ gm/yr, $R(0) = 1$ gm.

10. b. $R(t) = e^{-t/2337}$, so $R(-20) \approx 1.00859$ gm
 $R(200) \approx .91798$ gm.

11. a. $I'(x) = -kI(x)$.

11. b. The material affects the value of k . For a dense material like lead k is large—because then I decreases rapidly with x . For more “transparent” materials like air k is small.

11. c. Given $I(0) = I_0$, set $I(x) = I_0e^{-kx}$. Then

$$I'(x) = -k \cdot I_0e^{-kx} = -kI$$

and $I(0) = I_0e^0 = I_0$.

12. a. We know that $y = Ae^{kt}$ is a solution to the differential equation $y' = ky$. The crucial question is: Are there any *other* solutions? Suppose that $y = f(t)$ was such a solution. Then $f(0)$ has some value; let's call it Q . Then $y = f(t)$ is a solution to the initial value problem

$$y' = ky \quad y(0) = Q.$$

But $y = Qe^{kt}$ is *also* a solution. Since this initial value problem has a unique solution, we must have $f(t) = Qe^{kt}$. So $f(t)$ is *not* of a different form: it is just like the solutions $y = Ae^{kt}$ we've already identified.

Since the initial value problem

$$y' = ky \quad y(t_0) = C$$

involves the differential equation $y' = ky$, its solution is one of the (infinitely many) solutions to the differential equation—namely, $y = Ae^{kt}$ for some A .

12. b. We have $y(t_0) = Ae^{kt_0} = C$, so $A = Ce^{-kt_0}$ and thus

$$y(t) = Ce^{-kt_0}e^{kt} = Ce^{k(t-t_0)}.$$

Solving Other Differential Equations

13. a. Since $Q(t) - 20 = 70e^{-.1t}$, we have

$$Q'(t) = -.1 \cdot 70e^{-.1t} = -.1(Q - 20)$$

and $Q(0) = 70e^0 + 20 = 90$. Using the fact that $e = 2^{1/.69}$ we can convert the formula for Q —whose exponential function has base e —into the formula for T —whose exponential function has base 2.

13. b. The text should have $Q(0) = B$, not A . Also, this formula corresponds to the one in problem 13 (not problem 11, as the text says). Thus, since $Q - A = (B - A)e^{-kt}$,

$$Q'(t) = -k(B - A)e^{-kt} = -k(Q - A)$$

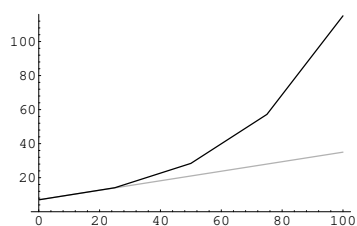
and $Q(0) = (B - A)e^0 + A = B$. To make this result correspond to exercise 13 in 4.2 we just have to replace e by $2^{1/.69}$, as in the previous part.

14. a. $P' = kP$, $F' = c$, where k and c are constants.

14. b. $P(t) = 7e^{.028t}$ million people, $F(t) = .28t + 7$ million units.

14. c. The last column tabulates the quantity investigated in the next part of this question.

t	$P(t)$	$F(t)$	$F(t)/P(t)$
0	7	7	1.00
25	14.0963	14	.99
50	28.3864	21	.74
75	57.1632	28	.49
100	115.1130	35	.30



14. d. As t increases $F(t)/P(t)$ will decrease, approaching zero. This supports Malthus's prediction that later generations of humans will not have enough food.

15. a. Since $e = 2^{1/.69}$, the modification we need looks like

$$v(t) = \frac{g}{b} (e^{-bt} - 1).$$

We must now show dv/dt and $-g - bv$ are the same, as functions of t . First, $v'(t) = -ge^{-bt}$; second,

$$-g - b \frac{g}{b} (e^{-bt} - 1) = -g - g(e^{-bt} - 1) = -ge^{-bt}.$$

Hence, the two expressions are the same.

15. b. The simplest thing to do is to modify the expression given in problem 22a, using $e = 2^{1/.69}$. This yields

$$x(t) = x_0 - \frac{g}{b}t - \frac{g}{b^2} (e^{-bt} - 1).$$

The student should then check that $dx/dt = v(t)$, where $v(t)$ is given in part a.

15. c. Using $b = .2$, we find $x(1) \approx 195.41$, so the object falls about 4.59 meters in the first second. Also, $x(2) \approx 182.77$, so the object falls about 17.23 meters in the first two seconds. In exercise 22, the corresponding values were 4.41 meters and 16.93 meters.

Interest Rates

16. $(1 + .18/12)^{12} \approx 1.195618$; that is, about 19.5618%.

17. $(1 + .18/365)^{365} \approx 1.1971642$; that is, about 19.71642%.

18. $(1 + .18/360)^{360} \approx 1.1971635$; that is, about 19.71635%. This first differs from the previous percentage in the sixth digit.

19. $(1 + .18/360)^{365} \approx 1.2001594$; that is, 20.01594%. This is noticeably larger than the previous computation.

20. The straightforward calculation is

$$(1 + .06/365)^{365} \approx 1.061831,$$

an effective annual rate of about 6.1831%. This is not what the bank claims as the effective rate. If we compound 1/360-th of the annual rate for 365 days, though, we do get the bank's value:

$$(1 + .06/360)^{365} \approx 1.062716,$$

an effective annual rate of about 6.2716%. This is .0885% more than the straightforward calculation.

21. 6% compounded quarterly yields

$$(1 + .06/4)^4 \approx 1.061364,$$

an effective annual rate of 6.1364%. 5.5% compounded continuously is

$$\lim_{n \rightarrow \infty} (1 + .055/n)^n = e^{.055} \approx 1.056541,$$

an effective annual rate of 5.6541%. The first account is better.

TEXT
ERRORS!

4.4 The Logarithm Function

- 1. a.** $1 + \ln(2) \approx 1.6931$ **1. i.** ≈ 2.3026
1. b. 3 **1. j.** $3 \ln(10) \approx 6.9078$
1. c. $1/e \approx .3679$ **1. k.** 10
1. d. $\frac{1}{2} \ln(e) = 1/2$ **1. l.** 1000
1. e. 2 **1. m.** $-\ln(e) = -1$
1. f. $2^3 = 8$ **1. n.** $-\ln(2) \approx -.6931$
1. g. $2^3 = 8$ **1. o.** $2^{-1} = 1/2$
1. h. $3^2 = 9$ **1. p.** $1/e^{3 \ln 2} = 1/8$

2. a. Euler's method yields

n	y
2^6	.6970687
2^8	.6941247
2^{10}	.6933914
2^{12}	.6932082
2^{14}	.6931624
2^{16}	.6931510
2^{18}	.6931481

We see that 3 decimal places have stabilized when $n = 2^{12}$. This means $\Delta x = 1/n = 1/2^{12} \approx .00025$.

2. b. Frankly, if we use the program already constructed to answer part a to calculate $\ln 6$, little extra work is involved. We get

n	y
2^6	1.8248057
2^8	1.7999284
2^{10}	1.7937959
2^{12}	1.7922682
2^{14}	1.7918866
2^{16}	1.7917912
2^{18}	1.7917674

Of course we can also use the fact that $\ln 6 = \ln 2 + \ln 3 = \ln 2 + \ln 3$, but then we still have to use the program to calculate $\ln 3$.

2. c. The real impact of the properties of logarithms can be seen in this question. If we use the program directly to calculate $\ln 1300$, the computed value after 2^{18} steps is 7.172597, and only the first two digits have stabilized. To get more digits, we must make more time-consuming calculations.

If, however, we write $1300 = 2^{10} \times 1300/2^{10}$, so that

$$\begin{aligned}
 \ln(1300) &= \ln(2^{10}) + \ln(1300/2^{10}) \\
 &= 10 \ln(2) + \ln(1300/2^{10}) \\
 &= 6.93147 \dots + \ln(1300/2^{10}),
 \end{aligned}$$

we can instead use the program to calculate $\ln(1300/2^{10})$. This calculation converges much more quickly:

n	y
2^6	.239095
2^8	.238760
2^{10}	.238676
2^{12}	.238654
2^{14}	.238649
2^{16}	.238648
2^{18}	.238648

Thus,

$$\ln(1300) = 6.93147 \dots + .23864 \dots = 7.1701 \dots$$

This approach gives more accuracy for a given amount of effort (i.e., for a given number of calculations).

3. Constant per capita growth implies the population is described by an exponential function: $P(t) = P_0 e^{kt}$. If we take $t = 0$ to be 1980, then $P_0 = 40$ million and $P(5) = 40e^{5k} = 45$ million. Hence $5k = \ln(45/40)$ so $k \approx .02356$. Therefore

$$P(t) = 40e^{.02356t} \text{ and } P(15) \approx 56.9560$$

million people.

4. a. $\ln(3x) = \ln 3 + \ln x$, so $(\ln(3x))' = 1/x$.

4. b. $17/x$.

4. c. $\ln(e^w) = w$, so $(\ln(e^w))' = 1$.

4. d. $\ln(2^t) = t \ln(2)$, so $(\ln(2^t))' = \ln(2) \approx .6931$.

4. e. $\pi \ln(3e^{4s}) = \pi(\ln(3) + 4s)$, so $(\pi \ln(3e^{4s}))' = 4\pi$.

5. We find the doubling time by solving the equation $400 = 200e^{.12t}$ (or $2 = e^{.12t}$) for t :

$$t = \frac{\ln(2)}{.12} \approx 5.7762 \text{ hours.}$$

To determine how long it takes the population to double again, we start with $P = 400$ so the equation to solve is $800 = 400e^{.12t}$. This also reduces to $2 = e^{.12t}$, so it has the same solution: $t = \ln(2)/.12 \approx 5.7762$ hours. For the third doubling, from 800 to 1600, we again have to solve $2 = e^{.12t}$. Thus, no matter what the *initial* population

size is, the doubling time is the same—namely, $\ln(2)/.12$ hours.

6. $s = .25$ at 3 inches, and $s = 1.5$ at 18 inches. Thus

$$R(.25) = Ae^{-.35 \times .25} \approx .9162A$$

$$R(1.5) = Ae^{-.35 \times 1.5} \approx .5916A.$$

To find where the intensity R is cut in half, we must solve the equation $A/2 = Ae^{-.35s}$ (or $1/2 = e^{-.35s}$) for s :

$$s = \ln(2)/.35 \approx 1.9804 \text{ feet}$$

As the rays continue on, after a further s feet their intensity is described by the formula $R(s) = Ae^{-.35s}/2$. To find where the intensity is $A/4$ we must solve the equation $A/4 = Ae^{-.35s}/2$ for s . This reduces to $1/2 = e^{-.35s}$, exactly as did the first equation. Hence its solution is the same: $s = \ln(2)/.35$ feet. In other words, the rays must travel an additional $\ln(2)/.35$ feet for their intensity to drop from $A/2$ to $A/4$.

It follows that, whenever the rays pass through $\ln(2)/.35$ feet of concrete, their intensity will be cut in half. It doesn't matter what the intensity is, or what part of the concrete the rays are in.

7. Let $y(t)$ denote the fraction of carbon-14 (C^{14}) remaining in an organism t years after it has died. It is given that y decays exponentially, so $y(t) = Ae^{-kt}$, for some positive constants A and k . Since $y(0) = 1$ (because the whole fraction of carbon-14 is present at the time of death), $A = 1$. Therefore, $y(t) = e^{-kt}$, and only k remains to be determined.

To determine k , use that fact that $.5 = e^{-5730k}$. Thus $k = \ln(.5)/5730 \approx 1.20968 \times 10^{-4}$.

Finally, if a bone has only 20% of its original carbon-14, then $y(t) = .2 = e^{-kt}$. Consequently,

$$t = \frac{-\ln(.2)}{k} \approx 13305 \text{ years};$$

the bone is about 13,300 years old.

8. We can suppose there were $P(t) = 2.5e^{kt}$ billion people in the world t years after 1960. Using the fact that $P(20) = 2.5e^{20k} = 3.5$ billion, we find

$$k = \frac{\ln(3.5/2.5)}{20} \approx .01682.$$

Therefore, in 2010 (when $t = 50$), we predict the world population will be

$$P(50) \approx 2.5e^{.01682 \times 50} \approx 5.7967,$$

or about 5.8 billion people. (Since the original data are given to only one decimal place, we report this value likewise to only one decimal place.)

9. The bacteria follow an exponential function: $y(t) = 1000e^{kt}$. We measure t in minutes. We are told the doubling time is 17 minutes (from 1000 to 2000), so $k = \ln(2)/17 \approx .04077$. To find how long it takes for y to reach 10000, we solve

$$10000 = 1000e^{kt}$$

for t : $t = \ln(10)/k \approx 56.47$, or about $56\frac{1}{2}$ minutes.

10. Suppose there are $y(t)$ lb of sugar undissolved after t hours. Since we assume sugar dissolves at a rate proportional to y , $y(t)$ is described by exponential decay: $y(t) = Ae^{-kt}$, for some positive constants A and k .

Since $40 = y(0) = Ae^{-0k} = A$, we can now write $y(t) = 40e^{-kt}$. Also, $12 = y(4) = 40e^{-4k}$, so

$$k = -\frac{\ln(12/40)}{4} \approx .3$$

If 99% is the sugar is dissolved, then 1% is undissolved. Since 1% of 40 lb is .4 lb, we want to solve for t in $.4 = 40e^{-.3t}$, or $.01 = e^{-.3t}$. Thus

$$t = \frac{\ln(.01)}{-.3} \approx 15.35 \text{ hours.}$$

11. Let $y(x)$ be the air pressure at a height of x thousand feet above sea level. We are told $y'(x)$ is proportional to y , so y is an exponential function of x : $y = Ae^{-kx}$, for some positive constants A and k . (We use a minus sign because pressure decreases with height.)

We are told $30 = y(0) = Ae^{-0k}$, so $A = 30$ and $y(x) = 30e^{-kx}$. Next, $24 = y(6) = 30e^{-6k}$, so $k = -\ln(24/30)/6 \approx .03719$. We want to find x for which $20 = y(x) = 30e^{-kx}$:

$$x = -\frac{\ln(20/30)}{k} \approx 10.902357,$$

or about 10,900 feet above sea level.

12. a. If A is the present value of \$8000 in 10 years time, when interest at 4% is compounded continuously, then

$$Ae^{.04 \times 10} = 8000, \quad A = \frac{8000}{e^4} = 5362.56.$$

Thus, the present value of the \$8000 is \$5362.56.

12. b. If the interest rate changes to 8%, then we solve

$$Ae^{.08 \times 10} = 8000, \quad A = \frac{8000}{e^8} = 3594.63.$$

At the higher interest rate, the present value of \$8000 drops to \$3594.63.

12. c. Now the interest rate is the unknown (let's call it k), and the equation we must solve is $8000 = 5000e^{10k}$. We find

$$k = \frac{\ln(8000/5000)}{10} = .047000363.$$

Thus, the necessary interest rate is about 4.7%.

13. All parts use the basic property $\ln(\exp(y)) = y$.

13. a. $1 = \exp(0)$, so $\ln(1) = \ln(\exp(0)) = 0$.

13. b. $m/n = \exp(\ln(m))/\exp(\ln(n)) = \exp(\ln(m)) \cdot \exp(-\ln(n)) = \exp(\ln(m) - \ln(n))$. Thus

$$\ln[m/n] = \ln[\exp(\ln(m) - \ln(n))] = \ln(m) - \ln(n).$$

13. c. $\ln[m^n] = \ln[(\exp(\ln(m)))^n] = \ln[\exp(n \ln(m))] = n \ln(m)$.

14. a.

Δx	$\Delta(\ln x)/\Delta x$
.1	.5004173
.01	.5000042
.001	.5000000

14. b.

Δx	$\Delta(e^x)/\Delta x$
.1	2.0033350
.01	2.0000333
.001	2.0000003

14. c. The two values are reciprocals. This agrees with the basic fact that the derivatives of inverse functions at corresponding points are reciprocals.

15. $f(x) = 3 \ln(x) + 2$.

16. a. $y(x) = a \ln(x) + b$.

16. b. $P(t) = 2 \ln(t) + 5$.

17. The domain of f is $x \geq 0$. The range is $f(x) \geq 0$. The domain of g is $x \geq 0$. The range is $g(x) \geq 0$.

17. a. The domain of f is $-\pi/2 \leq x \leq \pi/2$. The range is $-1 \leq f(x) \leq 1$. The domain of g is $-1 \leq x \leq 1$. The range is $-\pi/2 \leq g(x) \leq \pi/2$.

18. The inverse of f is the function g such that $f(g(x)) = x$ and $g(f(x)) = x$. Therefore, f will be its own inverse if $f(f(x)) = x$. In fact, $f(f(x)) = f(1/x) = 1/(1/x) = x$, so f is its own inverse.

The domain and range of f are all $x \neq 0$.

19. One inverse of f is $g(x) = x^{1/n}$. If n is an even integer, then the domain and range of f (and g) must be restricted to $0 \leq x$. If n is odd, there is no restriction on either the domain or the range.

20. a. Write $y = f(x) = 1 - 3x$ and solve for x : $x = (1 - y)/3 = g(y)$.

20. b. Since $f'(a) = -3$ (for all a) and $g'(b) = -1/3$ (for all b), we do have

$$g'(b) = \frac{1}{f'(a)},$$

so f and g satisfy Theorem 2.

21. Solve $y = f(x) = x^2 - 4$ for x : $x = \pm\sqrt{y+4}$. We therefore have two inverses (with different ranges):

$$g_+ = +\sqrt{y+4} \quad g_- = -\sqrt{y+4}$$

22. a. The inverse of $g(x) = \sqrt{x}$ is $f(x) = x^2$. Since $g(100) = 10$, we invoke Theorem 2 in the form

$$g'(100) = \frac{1}{f'(10)} = \frac{1}{2 \cdot 10} = .05.$$

22. b. The inverse of $g(x) = \arcsin(x)$ is $f(x) = \sin(x)$. Since $g(\sqrt{2}/2) = \pi/4$, we invoke Theorem 2 in the form

$$g'(\sqrt{2}/2) = \frac{1}{f'(\pi/4)} = \frac{1}{\cos(\pi/4)} = \frac{1}{\sqrt{2}/2} = \frac{2}{\sqrt{2}}.$$

23. a. Let $f(x) = x^2$; $g(x) = \sqrt{x}$. Then $b = f(a) = a^2$ and $a = \sqrt{b}$. By Theorem 2,

$$g'(b) = \frac{1}{f'(a)} = \frac{1}{2a} = \frac{1}{2\sqrt{b}}.$$

In other words, $g'(x) = 1/(2\sqrt{x})$.

23. b. Let $f(x) = x^n$; $g(x) = \sqrt[n]{x}$. Then $b = f(a) = a^n$ and $a = \sqrt[n]{b}$. Once again by Theorem 2,

$$g'(b) = \frac{1}{f'(a)} = \frac{1}{na^{n-1}} = \frac{1}{n(\sqrt[n]{b})^{n-1}}.$$

We can stop here, but we can also carry the last expression a little further. First of all,

$$(\sqrt[n]{b})^{n-1} = (b^{1/n})^{n-1} = b^{\frac{n-1}{n}} = b^{1-\frac{1}{n}}.$$

Then,

$$g'(b) = \frac{1}{n(\sqrt[n]{b})^{n-1}} = \frac{1}{n} \frac{1}{(\sqrt[n]{b})^{n-1}} = \frac{1}{n} b^{\frac{1}{n}-1}.$$

This demonstrates that when $g(x) = x^p$ and $p = 1/n$ (the reciprocal of a positive integer), then g' follows the standard pattern $g'(x) = px^{p-1}$ already seen in some other cases.

24. To compare growth rates, we look at the ratio e^x/b^x . Let

$$G = \frac{e^x}{b^x} = \left(\frac{e}{b}\right)^x = r^x.$$

If $e < b$ then $r < 1$ so $G = r^x \rightarrow 0$ as $x \rightarrow \infty$. Thus b^x grows faster than e^x .

If $1 < b < e$ then $r > 1$ so $G = r^x \rightarrow \infty$ as $x \rightarrow \infty$. Thus e^x grows faster than b^x .

4.5 The Equation $y' = f(t)$

1. An arbitrary constant C can (but need not be) added to any of the following answers.

1. a. $F(t) = \frac{5}{2}t^2 - 3t$.

1. b. $F(t) = \frac{1}{7}t^7 - \frac{4}{3}t^6 + 22\pi^3t$.

1. c. $F(t) = 5e^t + 3\cos(t)$.

1. d. $F(t) = 8t^{3/2}$.

1. e. $F(t) = \frac{1}{\ln 2}2^t - 7/8t^8$.

1. f. $F(t) = \frac{5}{4}e^{4t} - \ln t$.

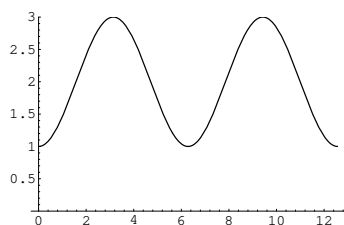
2. $y = G(x) = -1/x + 3.5$; $G(5) = 3.3$.

3. $y = F(x) = \ln(x) + 5$; $F(2) = \ln(2) + 5 \approx 5.69$.

4. $y = H(x) = \frac{1}{4}x^4 - \frac{7}{3}x^3 + 19x + \frac{257}{12}$; $H(3) = \frac{107}{3}$.

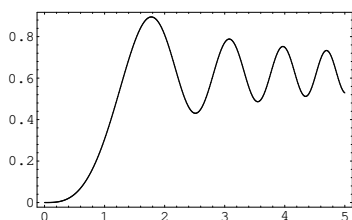
5. $y = L(x) = (e^{3x} - e^3)/3 + 6$; $L(-2) = (e^{-6} - e^3)/3 + 6 \approx -.69$.

6. a.



6. b. $y = 2 - \cos(x)$, so $y(2) = 2 - \cos(2) \approx 2.416$.

7. a.



7. b. $y'(0) = 0$ so the graph should be horizontal at $x = 0$. It is.

7. c. There are four local maxima. They happen at points where $y' = 0$. But $y' = \sin(x^2) = 0$ if $x^2 = 0, \pi, 2\pi$, etc. Thus $x = 0, \pm\sqrt{\pi}, \pm\sqrt{2\pi}, \pm\sqrt{3\pi}, \pm\sqrt{4\pi}$, and so forth. In fact, minima alternate with maxima, so the four visible maxima are at

$$x = \sqrt{\pi}, \sqrt{3\pi}, \sqrt{5\pi}, \sqrt{7\pi} = 1.77, 3.07, 3.96, 4.69$$

This fits with the graph.

7. d. The maximum value achieved is $y \approx 0.895$, which occurs when $x = \sqrt{\pi} \approx 1.7725$.

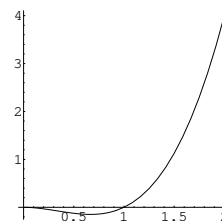
7. e.

n	y
2^8	.6500112
2^{10}	.6413604
2^{12}	.6391853
2^{14}	.6386408
2^{16}	.6385046
2^{18}	.6384705
2^{20}	.6384620

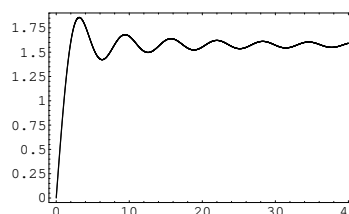
Thus, $y(6) \approx .638\dots$

8. a. If $dy/dt = 3t^2 - 2t$, then $y(t) = t^3 - t^2 + C$. As t increases from 0 to 1, $\Delta y = y(1) - y(0) = 0$. As t increases from 1 to 2, $\Delta y = y(2) - y(1) = 4$. As t increases from 0 to 4, $\Delta y = y(4) - y(0) = 4$.

8. b.



9. a.



9. b. There are six local maxima.

9. c. The maximum value achieved is $y \approx 1.85$. The maximum will occur at the first value of x for which $\sin(x)/x = 0$; that is, for $x = \pi$.

Solutions: Chapter 5

Techniques of Differentiation

5.1 The Differentiation Rules

Finding Derivatives

1. There are usually several ways to express the answer.

1. a. $15x^4 - 20x$.

1. b. $60x^{11}(\pi - \pi^2x^4) - 4\pi^2x^3(5x^{12} + 2)$.

1. c. $1/2\sqrt{u} + 9/u^4 + 14u^6$.

1. d. m .

1. e. $.5\cos x + \frac{1}{3}x^{-2/3}$.

1. f. $\frac{(5x^{12} + 2)(-4\pi^2x^3) - (\pi - \pi^2x^4)60x^{11}}{(5x^{12} + 2)^2}$.

1. g. $1/\sqrt{x} + \frac{1}{2}x^{-3/2}$.

1. h. $\sec^2 z(\sin z - 5) + \tan z \cos z$.

1. i. $(x^2 \cos x - 2x \sin x)/x^4$.

1. j. $2xe^x + e^x x^2$.

1. k. $e^x - \sin x$.

1. l. $\sec^2 x$.

1. m. $e^x \ln x + e^x/x$.

1. n. $\frac{(10 + \sin x) \ln(2) 2^x - 2^x \cos x}{(10 + \sin x)^2}$.

1. o. $\cos(e^x \cos x)(e^x \cos x - e^x \sin x)$.

1. p. $(-30 \sin t \sqrt[3]{t} e^{\cos t} - 10t^{-2/3} e^{\cos t})/25t^{2/3}$.

1. q. $(2x + xe^x + e^x)/(x^2 + xe^x)$.

1. r. $\frac{(7\sqrt{x} + 5)(10x + 1/x) - (5x^2 + \ln x)(7/2\sqrt{x})}{(7\sqrt{x} + 5)^2}$.

2. a. $f'(2) + g'(2) = 1$.

2. b. $5f'(2) - 2g'(2) = 12$.

2. c. $f'(2)g(2) + f(2)g'(2) = 5$.

2. d. $(g(2)f'(2) - f(2)g'(2))/(g(2))^2 = 11/16$.

2. e. $g'(3)f'(2) = 34$.

2. f. $g'(2)/2\sqrt{g(2)} = -1/4$.

2. g. $2 \cdot 2f(2) + 2^2f'(2) = 20$.

2. h. $2f(2)f'(2) + 2g(2)g'(2) = 4$.

2. i. $-f'(2)/(f(2))^2 = -2/9$.

2. j. $f'(3 \cdot 2 - g(1 + 2)^2)(3 - 2g(1 + 2)g'(1 + 2)) = f'(2) \cdot -65 = -130$.

2. k. Since the derivative of $f(g(t))$ at $t = 2$ is $f'(g(2))g'(2) = f'(4)g'(2)$, the additional information we would need is the value $f'(4)$.

2. l. Obtain the microscope equation for the function $h(t) = f(t)/g(t)$ at $t = 2$, and then apply it with $\Delta t = 1.95 - 2 = -.05$. From part d we see $h'(2) = 11/16$; therefore,

$$\Delta h \approx \frac{11}{16} \Delta t = \frac{11}{16} \times -.05 = -.034375.$$

Since $h(2) = 3/4 = .75$, we estimate $h(1.95) \approx .715625$.

3. a. One way to derive this is to set $g(t)h(t) = p(t)$. Then

$$(f(t)g(t)h(t))' = (f(t)p(t))' = f'(t)p(t) + f(t)p'(t),$$

and since $p'(t) = g'(t)h(t) + g(t)h'(t)$, we finally get

$$(f(t)g(t)h(t))' = f'(t)g(t)h(t) + f(t)g'(t)h(t) + f(t)g(t)h'(t).$$

3. b. Let f , g , and h denote the length, width, and height of the box, respectively. Then the volume is $f \cdot g \cdot h$. If $t = 0$ is the moment in question, then

$$\begin{aligned} f(0) &= g(0) = h(0) = 10 \\ f'(0) &= 3, \quad g'(0) = 6, \quad h'(0) = -5 \end{aligned}$$

From this information and part **a**, we see the volume is growing at the rate

$$\begin{aligned}(f(0)g(0)h(0))' &= \\ &= f'(0)g(0)h(0) + f(0)g'(0)h(0) + f(0)g(0)h'(0) \\ &= 300 + 600 - 500 = 400 \text{ inches/minute.}\end{aligned}$$

3. c. This time we know

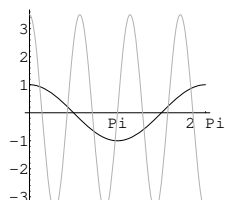
$$\begin{aligned}f(0) &= 10, g(0) = 12, h(0) = 8 \\ f'(0) &= 3, h'(0) = -2,\end{aligned}$$

and we must determine $g'(0)$ under the assumption that the derivative of the volume is 0:

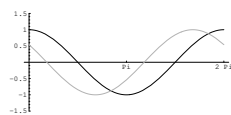
$$\begin{aligned}0 &= f'(0)g(0)h(0) + f(0)g'(0)h(0) + f(0)g(0)h'(0) \\ &= 288 + 80g'(0) - 240\end{aligned}$$

Hence $g'(0) = -3/5$ inches/minute.

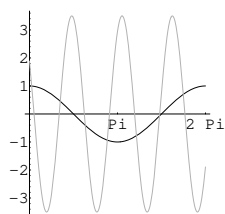
4. a. $g'(x) = mf'(mx)$. At the right, f' is black and g' is gray. In comparison to the graph of f' , the graph of g' is stretched vertically by a factor of m but *compressed* horizontally by that factor.



4. b. $g'(x) = f'(x+b)$. At the right, f' is black and g' is gray. In comparison to the graph of f' , the graph of g' is shifted horizontally to the left by the amount b (to the right if b is negative).



4. c. $g'(x) = mf'(mx+b)$. At the right, f' is black and g' is gray. In comparison to the graph of f' , the graph of g' is stretched vertically by a factor of m , compressed horizontally by m , and shifted to the left by b .



5. The derivative of $-1/x$ is $1/x^2$, and this is always positive. None of the other functions has a derivative which is always positive. (Compare exercise 5 of section 3.3.)

6. a. The units are cm^3/cm , which reduces to (but does not have the same meaning as!) cm^2 .

6. b. A derivative is a *rate*, so it is most appropriate that its units reflect that fact. Since V is a volume and r is a

length, the units for the derivative $V'(r)$ should be volume units per linear unit, or cm^3/cm . The unit cm^2 does not reveal that a rate is involved.

7. a. First,

$$\left(\frac{1}{1-x^2}\right)' = \frac{(1-x^2) \cdot 0 - 1 \cdot -2x}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}$$

Next,

$$\left(\frac{x^2}{1-x^2}\right)' = \frac{(1-x^2)2x - x^2 \cdot -2x}{(1-x^2)^2} = \frac{2x}{(1-x^2)^2}$$

7. b. Since $(f(x) - g(x))' = 0$, $f(x) - g(x)$ must be a constant. In other words, f and g differ by a constant.

7. c. Since $\frac{1}{1-x^2} = \frac{x^2}{1-x^2} + C$, we have

$$C = \frac{1}{1-x^2} - \frac{x^2}{1-x^2} = \frac{1-x^2}{1-x^2} = 1.$$

8. Let E be total daily energy consumption, P the current population. Then E/P is the daily *per capita* energy consumption. Its current value is $(16 \times 10^{13})/(2 \times 10^8) = 8 \times 10^5$ BTU per person. Its rate of change is

$$\begin{aligned}\left(\frac{E}{P}\right)' &= \frac{PE' - EP'}{P^2} \\ &= \frac{(2 \times 10^8)(6 \times 10^{11}) - (16 \times 10^{13})(10^6)}{(2 \times 10^8)^2} \\ &= \frac{12 \times 10^{19} - 16 \times 10^{19}}{4 \times 10^{16}} \\ &= -10^3 \text{ BTU/person per year.}\end{aligned}$$

Thus, daily per capita energy consumption is *falling* at the rate of 1000 BTU/person per year. In other words, one year from now each person's *daily* consumption will be 1000 BTU less than it is now.

A fraction (like E/P) can fall even when the numerator and denominator are both rising because the denominator can be rising faster than the numerator. That's what is happening here.

9. Let P be the population and C the yearly per capita energy expenditure. Then PC is the *total* expenditure. The current total is $(15 \times 10^6)(10^3) = 15 \times 10^9$ dollars. It is growing at the rate

$$\begin{aligned}(PC)' &= PC' + P'C \\ &= (15 \times 10^6)(8) + (10^4)(10^3),\end{aligned}$$

or 13×10^7 dollars per year.

10. Let P be the population and T the total income. Then T/P is the per capita income. Its current value is $(20 \times 10^9)/(30 \times 10^6) = \frac{2}{3} \times 10^3$, or 667 dollars. Its rate of change is

$$\begin{aligned}\left(\frac{T}{P}\right)' &= \frac{PT' - TP'}{P^2} \\ &= \frac{(30 \times 10^6)(500 \times 10^6) - (20 \times 10^9)(4 \times 10^3)}{(30 \times 10^6)^2} \\ &= 16.58 \text{ dollars per person per year.}\end{aligned}$$

11. Let s be the length of the side; then $A = s^2$ is its area. Since $s = 100$ feet and $s' = -2$ feet per day, $A' = 2ss' = -400$ square feet per day.

If $s' = -2$ is maintained for 5 days, then $\Delta s = -10$ feet. Hence, after 5 days the top will be a square whose side is $s = 90$ feet long. The area of the top will be shrinking at the rate of $A' = 2ss' = -360$ square feet per day.

12. The volume is $V = s^3$ cubic feet, and it is shrinking at the rate $V' = 3s^2s' = -60000$ cubic feet per day. After five days, $s = 90$ so the iceberg will then be shrinking at the rate $V' = -48600$ cubic feet per day.

Deriving Differentiation Rules

13. a. One way to get all 16 terms is to construct various products by choosing either x or Δx from the first factor, x or Δx from the second factor, and so on. Here is a table that does this:

first	second	third	fourth	product
x	x	x	x	x^4
Δx	x	x	x	$x^3\Delta x$
x	Δx	x	x	$x^3\Delta x$
x	x	Δx	x	$x^3\Delta x$
x	x	x	Δx	$x^3\Delta x$
Δx	Δx	x	x	$x^2(\Delta x)^2$
Δx	x	Δx	x	$x^2(\Delta x)^2$
Δx	x	x	Δx	$x^2(\Delta x)^2$
x	Δx	Δx	x	$x^2(\Delta x)^2$
x	Δx	x	Δx	$x^2(\Delta x)^2$
x	x	Δx	Δx	$x^2(\Delta x)^2$
x	Δx	Δx	Δx	$x(\Delta x)^3$
Δx	x	Δx	Δx	$x(\Delta x)^3$
Δx	Δx	x	Δx	$x(\Delta x)^3$
Δx	Δx	Δx	x	$x(\Delta x)^3$
Δx	Δx	Δx	Δx	$(\Delta x)^4$

$(x + \Delta x)^4$ is the sum of all the terms in the last column.

13. b. Only one term involves no Δx ; it has the form x^4 .

13. c. Four terms involve a single Δx ; they have the form $x^3\Delta x$.

13. d. If we write $f(x + \Delta x) = Ax^4 + B\Delta x + R(\Delta x)^2$, then

$$A = 1, B = 4x^3, R = 6x^2 + 4x\Delta x + (\Delta x)^2 \cdot d$$

13. e. Since $f(x) = x^4$,

$$\begin{aligned}\frac{f(x + \Delta x) - f(x)}{\Delta x} &= \frac{x^4 + 4x^3\Delta x + R(\Delta x)^2 - x^4}{\Delta x} \\ &= 4x^3 + R\Delta x.\end{aligned}$$

13. f. As $\Delta x \rightarrow 0$, $R \rightarrow 6x^2$ and $R\Delta x \rightarrow 0$; hence

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = 4x^3.$$

This is compatible with the general rule for the derivative of x^n .

14. a. We carry out only ii:

$$\begin{aligned}(a + b)(\underline{a} + \underline{b})(A + B) &= \\ \underline{a}\underline{a}A + \underline{a}\underline{b}A + \underline{b}\underline{a}A + \underline{b}\underline{b}A + \underline{a}\underline{a}B + \underline{a}\underline{b}B + \underline{b}\underline{a}B + \underline{b}\underline{b}B.\end{aligned}$$

There are $2 \times 2 \times 2$ ways to make a choice consisting of one entry from the first, one entry from the second, and one from the third.

In the general case, where we are choosing either x or Δx from each of n factors, there is only *one* way to choose only xs . The resulting product is x^n .

There are n ways to make the choices so that exactly one Δ appears: Δx can be chosen from the first factor, or from the second, or from the third, and so on. When one Δx is chosen, the remaining $n - 1$ choices are all xs . The resulting products therefore all have the form $x^{n-1}\Delta x$.

In every other product, Δx must be chosen at least twice. There are many such terms, but all of them have the form $Q(\Delta x)^2$; Q will involve additional x factors, and possibly Δx factors as well.

It follows that

$$(x + \Delta x)^n = x^n + nx^{n-1}\Delta x + R(\Delta x)^2,$$

where R is a polynomial expression in x and Δx . (The exact form of R is found in the last problem for $n = 4$.)

$$\mathbf{14. b.} \quad \frac{f(x + \Delta x) - f(x)}{\Delta x} = nx^{n-1} + R\Delta x.$$

$$\mathbf{14. c.} \quad \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = nx^{n-1}.$$

15. We have

$$\begin{aligned}(x^r)' &= (e^{r \ln(x)})' = e^{r \ln(x)} (r \ln(x))' \\ &= e^{r \ln(x)} \cdot \frac{r}{x} = x^r \cdot \frac{r}{x} = rx^{r-1}.\end{aligned}$$

16. According to the rule, $(x^r)' = rx^{r-1} = 0 \cdot x^{-1} = 0$ when $r = 0$. But when $r = 0$, $x^r = 1$, a constant function, so its derivative *should* be 0. The rule does hold.

17. a. We just need to rearrange terms:

$$\begin{aligned}F(x + \Delta x) - F(x) &= (f(x + \Delta x) + g(x + \Delta x)) - (f(x) + g(x)) \\ &= f(x + \Delta x) - f(x) + g(x + \Delta x) - g(x).\end{aligned}$$

17. b. Therefore $F'(x)$ is

$$\begin{aligned}\lim_{\Delta x \rightarrow 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x) + g(x + \Delta x) - g(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \rightarrow 0} \frac{g(x + \Delta x) - g(x)}{\Delta x} \\ &= f'(x) + g'(x)\end{aligned}$$

18. a. $f'(x) = F'(x)g(x) + g'(x)F(x).$

18. b. $f'(x) = F'(x)g(x) + g'(x)\frac{f(x)}{g(x)}.$

18. c. We have $f'(x)g(x) = F'(x)(g(x))^2 + g'(x)f(x)$, so

$$F'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}.$$

19. a. Let $n = -m$, so $f(x) = x^n = 1/x^m$. By the quotient rule,

$$\left(\frac{1}{x^m}\right)' = \frac{0 \cdot x^m - mx^{m-1}}{(x^m)^2} = -mx^{-m-1}.$$

19. b. If we replace $-m$ by n , then $f'(x) = -mx^{-m-1} = nx^{n-1}.$

20. a. $f(x + \Delta x) = \sin x \cos \Delta x + \cos x \sin \Delta x.$

20. b. $P(\Delta x) = \frac{\cos \Delta x - 1}{\Delta x}, \quad Q(\Delta x) = \frac{\sin \Delta x}{x}.$

20. c. $\lim_{\Delta x \rightarrow 0} P(\Delta x) = 0, \lim_{\Delta x \rightarrow 0} Q(\Delta x) = 1.$

20. d. $f'(x) = 0 \cdot \sin x + 1 \cdot \cos x = \cos x.$

20. e. If $f(x) = \cos x$, then

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\cos x \cos \Delta x - \sin x \sin \Delta x - \cos x}{\Delta x}.$$

If we again write this as $P \sin x + Q \cos x$, then this time

$$P = -\frac{\sin \Delta x}{\Delta x} \rightarrow -1, \quad Q = \frac{\cos \Delta x - 1}{\Delta x} \rightarrow 0.$$

Consequently, $f'(x) = -\sin x.$

21. a. $f(x) = \tan x = \sin x / \cos x$, so

$$\begin{aligned}f'(x) &= \frac{\cos x(\sin x)' - \sin x(\cos x)'}{(\cos x)^2} \\ &= \frac{(\cos x)^2 + (\sin x)^2}{(\cos x)^2} = \frac{1}{(\cos x)^2} = \sec^2 x.\end{aligned}$$

21. b. $f(x) = \csc x = 1/\sin x$, so

$$\begin{aligned}f'(x) &= \frac{0 - (\sin x)'}{(\sin x)^2} = \frac{-\cos x}{(\sin x)^2} \\ &= -\frac{\cos x}{\sin x} \cdot \frac{1}{\sin x} = -\cot x \csc x.\end{aligned}$$

21. c. $f(x) = \sec x = 1/\cos x$, so

$$\begin{aligned}f'(x) &= \frac{0 - (\cos x)'}{(\cos x)^2} = \frac{\sin x}{(\cos x)^2} \\ &= \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x} = \tan x \sec x.\end{aligned}$$

21. d. If $f(x) = 1/\tan x$ (rather than $\cos x/\sin x$),

$$\begin{aligned}f'(x) &= \frac{0 - (\tan x)'}{(\tan x)^2} = -\frac{(\sec x)^2}{(\tan x)^2} \\ &= -\frac{1}{(\cos x)^2} \cdot \frac{(\cos x)^2}{(\sin x)^2} = -\csc^2 x.\end{aligned}$$

Differential Equations

22. a. $f' = 3e^{3x-2}, f'' = 9e^{3x-2}.$

22. b. $f' = \omega \cos \omega x, f'' = -\omega^2 \sin \omega x.$

22. c. $f' = 2xe^x + x^2e^x = (2x + x^2)e^x,$
 $f'' = (2 + 2x)e^x + (2x + x^2)e^x = (2 + 4x + x^2)e^x.$

23. To show that e^{3x} and e^{-3x} satisfy the differential equation, it is sufficient to show that $g(x) = \alpha e^{3x} + \beta e^{-3x}$ satisfies the equation, for any α and β :

$$g' = 3\alpha e^{3x} - 3\beta e^{-3x}, \quad g'' = 9\alpha e^{3x} + 9\beta e^{-3x} = 9g,$$

so g *does* satisfy the differential equation $y'' = 9y$. Since

$$g(0) = \alpha + \beta = 12, \quad g'(0) = 3\alpha - 3\beta = 15,$$

we find $\alpha = 17/2$, $\beta = 7/2$.

24. This is similar to the previous question. If $y = a \sin x + b \cos x$, then

$$y' = a \cos x - b \sin x, \quad y'' = -a \sin x - b \cos x = -y.$$

Therefore $y'' + y = 0$, for any choice of a and b .

We look for a function $g(x) = a \sin x + b \cos x$ satisfying

$$g(0) = b = 1, \quad g'(0) = a = 4.$$

Thus $g(x) = 4 \sin x + \cos x$.

25. If $y = \sin \omega x$, then from exercise 22b we know $y'' = -\omega^2 y$, so $y'' + \omega^2 y = 0$. Any function

$$y = a \sin \omega x + b \cos \omega x$$

is also a solution.

To solve the last part of the problem, we look for a function $L = a \sin \omega x + b \cos \omega x$. Since $L'' + 4L = 0$, we should take $\omega^2 = 4$, or $\omega = 2$. Then

$$L(0) = b = 36, \quad L'(0) = \omega a = 64,$$

so $a = 32$ and $L(x) = 32 \sin 2x + 36 \cos 2x$.

The Colorado River Problem

26. a. Suppose the lake is $w(t)$ yards wide after t weeks, and $l(t)$ yards long. The area now is $A(0) = 950 \times 50 = 47500$ square yards.

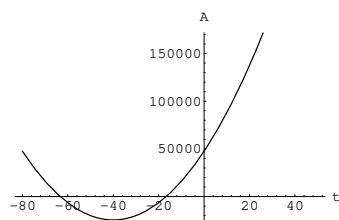
26. b. The area is growing at the rate

$$A' = w'l + wl' = (3)(950) + (50)(15) = 3600$$

square yards per week.

27. a. Since $w'(t) = 3$, $w(t) = 3t + 50$. Since $l'(t) = 15$, $l(t) = 15t + 950$. Therefore

$$A(t) = (3t + 50)(15t + 950) = 45t^2 + 3600t + 47500.$$



27. b. The lake will cover 30 acres when $145,200 = 45t^2 + 3600t + 47500$, or $45t^2 + 3600t - 97700 = 0$. By the quadratic formula,

$$t = \frac{-3600 + \sqrt{3600^2 + 4 \cdot 45 \cdot 97700}}{90} \approx 21.4094$$

weeks. We can disregard the negative root.

27. c. $A'(t) = 90t + 3600$, so $A'(21.4094) = 5526.8436$ square yards per week.

28. The rates are not the same. When the lake is 145,200 yd² the area of the lake is increasing 1926.85 yd²/week more than when the lake was 47,500 yd². Even though w' and l' are constant, A' is *not* constant, because $A' = w'l + wl'$ and w and l do not remain constant.

29. We have $E'(t) = .22A(t)$ cubic yards per week.

30. Water flows into the reservoir at the rate 25000 cubic yards per week. It evaporates at the rate E' . To find when the inflow and evaporation rates exactly balance, we must set

$$25000 = E' = .22(45t^2 + 3600t + 47500)$$

(and then solve $9.9t^2 + 792t - 14550 = 0$). This gives $t \approx 15.4$ weeks when the lake stops filling.

5.2 Finding Partial Derivatives

1. a. $f_x = 2xy$, $f_y = x^2$.

1. b. $f_x = 1/2\sqrt{x+y}$, $f_y = 1/2\sqrt{x+y}$.

1. c. $f_x = ye^{xy}$, $f_y = xe^{xy}$.

1. d. $f_x = -y/x^2$, $f_y = 1/x$.

1. e. $f_x = 1/(y+z)$, $f_y = (z-x)/(y+z)^2$,
 $f_z = -(x+y)/(y+z)^2$.

1. f. $f_x = (-y/x^2) \cos(y/x)$, $f_y = (1/x) \cos(y/x)$.

2. a. $f_x = -e^{-(x+2y)}(2x-5y) + 2e^{-(x+2y)}$,
 $f_y = -2e^{-(x+2y)}(2x-5y) - 5e^{-(x+2y)}$.

2. b. Solve $0 = f_x = (-2x+5y+2)e^{-(x+2y)}$. At any point on the line $-2x+5y+2=0$, we have $f_x = 0$.

2. c. Here we want $f_x = -f_y$. This will be true if

$$-2x+5y+2 = -4x+10y-5 \quad \text{or} \quad 2x-5y+7=0.$$

This is another line, parallel to the one in the previous part.

$$3. \quad g_u = \frac{(1+u^2+v^4)(\cos u + 7v) - 2u(\sin u + v^2 + 7uv)}{(1+u^2+v^4)^2},$$

$$g_v = \frac{(1+u^2+v^4)(2v+7u) - 4v^3(\sin u + v^2 + 7uv)}{(1+u^2+v^4)^2}.$$

$$4. \text{ a. } f_{xx} = 2y, f_{xy} = 2x = f_{yx}, f_{yy} = 0.$$

$$4. \text{ b. } f_{xx} = -1/4(x+y)^{3/2}, f_{xy} = -1/4(x+y)^{3/2} = f_{yx}, f_{yy} = -1/4(x+y)^{3/2}.$$

$$4. \text{ c. } f_{xx} = y^2 e^{xy}, f_{xy} = e^{xy} + xy e^{xy} = f_{yx}, f_{yy} = x^2 e^{xy}.$$

$$4. \text{ d. } f_{xx} = 2y/x^3, f_{xy} = -x^{-2} = f_{yx}, f_{yy} = 0.$$

$$4. \text{ e. } f_{xx} = (2xy \cos(y/x) - y^2 \sin(y/x))/x^4, f_{xy} = (y \sin(y/x) - x \cos(y/x))/x^3 = f_{yx}, f_{yy} = -\sin(y/x)/x^2.$$

Eradication of Disease

5. First, we see what p is before and after the public health efforts. Before: $p = 96.30\%$ of the population must be vaccinated. After: $p = 94.17\%$ must be vaccinated. This is a *decrease* of 2.13%.

Second, we try to see what factors play the greatest role in causing this decrease. We use the microscope equation:

$$L: \quad \Delta p \approx p_L \Delta L = .07\% \times 5 = .35\%$$

$$A: \quad \Delta p \approx p_A \Delta A = -1.78\% \times 1 = -1.78\%$$

$$V: \quad \Delta p \approx p_V \Delta V = 1.85\% \times -.5 = -.93\%$$

Thus, increasing the age at which children catch measles has had the greatest effect on reducing p . The decrease in the age of vaccination has been only about half as effective. The increase in life expectancy actually tends to push p up, not down.

By the way, the total of the three percentages (which is -2.36%) does not equal the actual change (which is -2.13%), because the microscope equation is only an approximation.

Partial Differential Equations

6. On the one hand,

$$z_x = \frac{-x}{2t^{3/2}} \exp \frac{-x^2}{4t} \quad \text{and}$$

$$z_{xx} = \frac{-1}{2t^{3/2}} \exp \frac{-x^2}{4t} + \frac{x^2}{4t^{5/2}} \exp \frac{-x^2}{4t}$$

$$= \frac{(x^2 - 2t)}{4t^{5/2}} \exp \frac{-x^2}{4t};$$

on the other,

$$z_t = \frac{-1}{2t^{3/2}} \exp \frac{-x^2}{4t} + \frac{1}{t^{1/2}} \frac{x^2}{4t^2} \exp \frac{-x^2}{4t}$$

$$= \frac{(x^2 - 2t)}{4t^{5/2}} \exp \frac{-x^2}{4t} = z_{xx}.$$

$$7. \quad z_x = p, z_{xx} = 0, z_y = q, z_{yy} = 0. \text{ Therefore, } z_{xx} + z_{yy} = 0 + 0 = 0.$$

$$8. \quad z_x = e^x \sin y, z_{xx} = e^x \sin y. \quad z_y = e^x \cos y, z_{yy} = -e^x \sin y. \text{ Therefore, } z_{xx} + z_{yy} = \sin y e^x - \sin y e^x = 0.$$

5.3 The Shape of the Graph of a Function

Describing Functions

1. a. Continuous but not differentiable.

1. b. Continuous and differentiable.

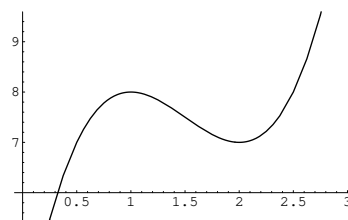
1. c. Not continuous and not differentiable.

1. d. Not continuous and not differentiable.

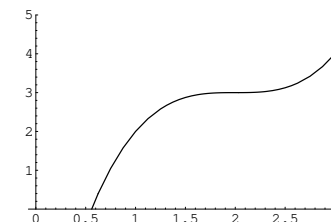
2. part	sign f'	f'	sign f''
a	+	$\nearrow$	+
b	0	$\searrow$	-
c	-	$\searrow$	-
d	-	$\nearrow$	+
e	+	$\searrow$	-
f	0	$\nearrow$	+

3. The following graphs are merely examples; there are many other possibilities.

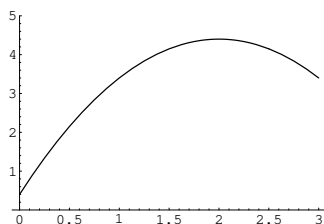
3. a.



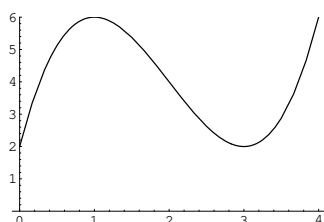
3. b.



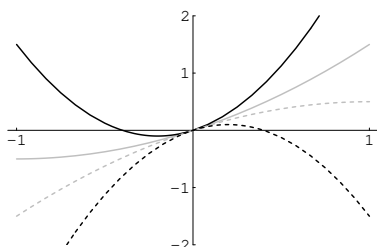
3. c.



3. d.



4. a.

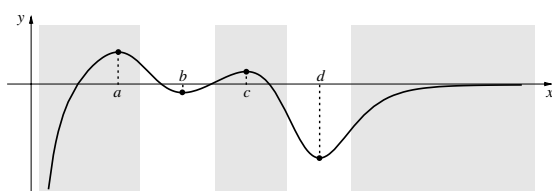


Key: f solid gray h dashed gray
 g solid black k dashed black

4. b. The magnitude of the second derivative tells how curved the graph is. "The greater the value of f'' , the more sharply the graph of f is curved upward."

5. a. The second derivative is greater at d , because the graph is curving more sharply there than at b .

5. b.

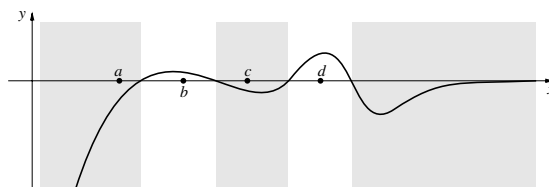


In the shaded intervals, the graph is concave down; elsewhere, it is concave up.

5. c. At the points where the graph changes concavity, the second derivative changes sign. Consequently, the value of the second derivative must be 0 at those points.

5. d. Near the right-hand end of the graph, the second derivative must be approaching zero. This is because the graph itself is becoming straight as it approaches the axis, and a straight graph has a zero second derivative.

5. e. Any graph with the same qualitative features as the following is acceptable as a description of the second derivative. There are three essential features to note: (1) the graph should be negative in the shaded regions and positive elsewhere; (2) the graph should be higher at d than at b ; (3) the graph should approach the x axis for large x .



6. First suppose $f'(c) = 0$ and $f''(c) > 0$. The second statement tells us that f' is *increasing* near $x = c$. But $f' = 0$ at $x = c$, so f' must be negative to the left of c and positive to the right of c . So the graph of f itself must slope *down* on the left of c and *up* on the right of c . Hence f must have a **minimum** exactly at $x = c$.

If we suppose instead that $f''(c) < 0$, then f' must be *decreasing* near $x = c$. Since $f' = 0$ at $x = c$, f' must be positive to the left of c and negative to the right. So the graph of f must slope *up* on the left of c and *down* on the right of c . Hence f must have a **maximum** exactly at $x = c$.

Finding Critical Points

7. a. $f' = \frac{1}{3}x^{-2/3}$ so f' is undefined at $x = 0$; this is the only critical point. The graph has a vertical tangent at $x = 0$.

7. b. $f' = 3x^2 + 3x - 6 = 3(x+2)(x-1)$; $x = -2, 1$ are the critical points. The first is a local maximum, the second a local minimum.

7. c. $f' = -3(x-1)^{-2}$ so f is undefined at $x = 1$. But f itself is undefined at $x = 1$, so this is *not* a critical point.

7. d. $f' = \cos x$; for every integer n , $x = (2n+1)\pi/2$ is a critical point. Maxima and minima alternate, with $x = \pi/2$ having a maximum.

7. e. $f' = -x/\sqrt{1-x^2}$; f' is undefined for $x = \pm 1$ (while f is defined there) and $f' = 0$ for $x = 0$, so 0 and ± 1 are the critical points. There is a maximum at $x = 0$ and minima at $x = \pm 1$.

7. f. $f' = e^x(x-1)/x^2$; f' is undefined for $x = 0$ but so is f ; $f' = 0$ for $x = 1$, so 1 is the only critical point. It is a minimum.

7. g. $f' = \ln x + 1$; $f' = 0$ for $x = e^{-1} \approx .3679$, and this is the only critical point. It is a minimum.

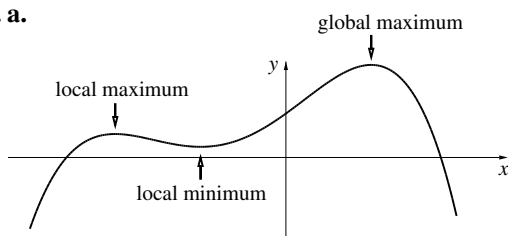
7. h. If we write f in the form $x^c + x^{-c}$, then we see that replacing c by $-c$ doesn't change the expression. Therefore we can restrict c to positive values.

Since $f(x)$ is undefined at $x = 0$, we will not consider 0 to be a critical point of f . Since $f' = c(x^{2c} - 1)/x^{c+1}$, we see $f' = 0$ when $x^{2c} = 1$, or

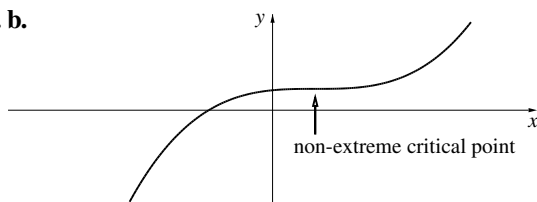
$$x = \sqrt[2c]{1} = \begin{cases} \pm 1 & \text{if } c \text{ is an integer} \\ +1 & \text{otherwise} \end{cases}$$

Thus, 1 is a critical point and -1 will also be one when c is an integer. In all cases, $x = 1$ is a minimum. When c is even, $f(-1) = f(1)$ and $x = -1$ is a minimum, too; when c is odd, $f(-1) = -f(1)$ and $x = -1$ is a maximum.

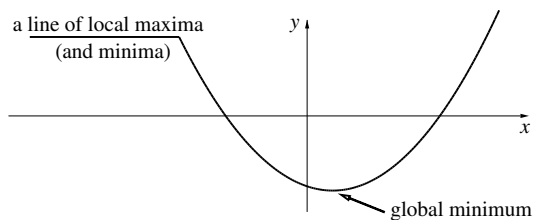
8. a.



8. b.



8. c.

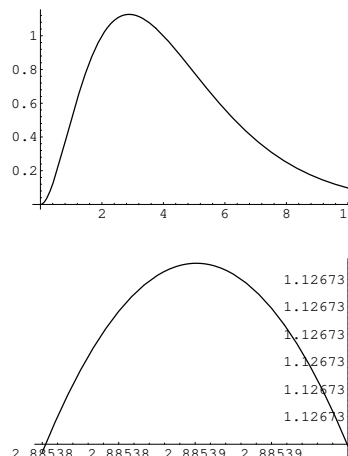


Finding Extremes

9. Method: Find point where $f' = 0$. In fact, we have $f' = 1 - 7/x^2$, so $f' = 0$ when $x = \pm\sqrt{7}$. We are restricted to $x > 0$, so $x = +\sqrt{7}$ is the critical point. Since $f'' = 14/x^3$, and this is positive for $x > 0$, the critical point $x = \sqrt{7} \approx 2.6458$ is a **minimum**.

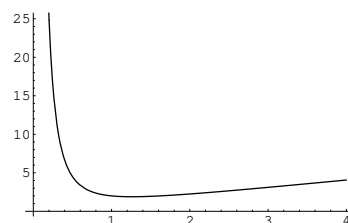
10. Method: Inspect the graph of f on the interval $[1, 2]$. This shows f is decreasing, a fact which is confirmed by noting $f' = 1 - 7/x^2 < 0$ on this interval. Hence the minimum is at the right boundary point $x = 2$.

11.



The second graph is a magnification of the first. Together they show a maximum occurs where $x \approx 2.8854$. Also, $f' = 2x2^{-x} - x^2 \ln(2)2^{-x} = x2^{-x}(2 - x \ln(2))$. Therefore, $f' = 0$ if $2 - x \ln(2) = 0$, or $x = 2/\ln(2) = 2.8853900817... = 2.8853901$ to seven decimal places accuracy.

12.



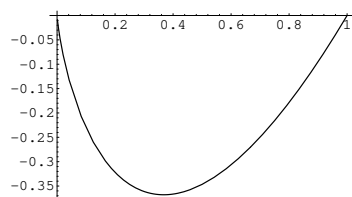
From the graph it appears that y is smallest when $x = 1.25$. Also, $y' = 1 - 2/x^3 = 0$ for $x = \sqrt[3]{2} \approx 1.25992$.

13. Method: the graph of $y = x + 4/x^2$ suggests that the smallest value of y occurs near $x = 2$. Calculating $y' = 1 - 8/x^3$, we find $y' = 0$ exactly at $x = 2$. As a further check that this is a minimum, note that $y'' = 24/x^4 > 0$ so the graph of y is concave up everywhere. This implies that when $x > 0$, y has its *global* minimum at $x = 2$.

14. $y' = 4x^3 - 84x - 80 = 4(x^3 - 21x - 20)$. Notice that $x = -1$ is a root; hence $x + 1$ is a factor. Dividing $x^3 - 21x - 20$ by $x + 1$, we get $x^2 - x - 20 = (x - 5)(x + 4)$. Thus $y' = 4(x + 1)(x - 5)(x + 4)$ and $y' = 0$ for $x = -4, -1, 5$. The local minima of y occur at $x = -4$ and $x = 5$ (because $y'' = 12(x^2 - 7)$ is positive for those values of x). Since $y(-4) = -96$ and $y(5) = -825$, the global minimum is -825 and it occurs at $x = 5$.

15. $y' = 4x^3 - 12x = 4x(x^2 - 3)$. Therefore $y' = 0$ when $x = 0$ and $x = \pm\sqrt{3}$. The minima are at $\pm\sqrt{3}$. Since $y(\pm\sqrt{3}) = -2$, this is the global minimum and it is achieved at both points $x = \pm\sqrt{3}$.

16.



According to the graph, y *does* have a minimum. It occurs where $y' = \ln x + 1 = 0$, that is, at $x = e^{-1} \approx .3679$.

17. The question can be rephrased as asking for the maximum value of $y = x^2 - x^3$. Since $y' = 2x - 3x^2 = x(2 - 3x)$, the maximum will occur when $x = 2/3$. (The second derivative $y'' = 2 - 6x$ is negative here.) The value of y is $(\frac{2}{3})^2 - (\frac{2}{3})^3 = \frac{4}{27}$.

18. When $0 < p < q$, $y = x^p - x^q$ is positive only when $0 < x < 1$, so we can restrict ourselves to this domain. Since $y' = px^{p-1} - qx^{q-1} = x^{p-1}(p - qx^{q-p})$, the maximum occurs where $x = \sqrt[q-p]{p/q}$. Therefore, the maximum value is

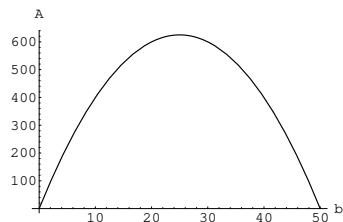
$$y = \left(\frac{p}{q}\right)^{\frac{p}{q-p}} - \left(\frac{p}{q}\right)^{\frac{q}{q-p}}.$$

5.4 Optimal Shapes

1. Suppose b and h are the base and height of the rectangle, respectively. The perimeter is fixed: $2b + 2h = P$, so we can express b in terms of h : $b = (P - 2h)/2$. We want to maximize the area $A = bh$. Because b depends on h , A is a function of h alone: $A = h(P - 2h)/2 = \frac{1}{2}hP - h^2$. To find the maximum, we set $A' = 0$. Thus,

$$A' = \frac{P}{2} - 2h = 0 \quad \text{if} \quad h = \frac{P}{4}.$$

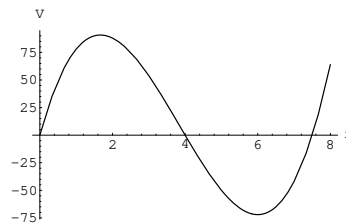
This is a maximum because $A''(P/4) = -2 < 0$. This implies $b = P/4$ as well, so the optimal rectangle is a square.



2. Suppose the sides of the corner squares measure x inches. Then, when we fold up the sides to make a box, the base of the box measures $8 - 2x$ inches by $15 - 2x$ inches, and its volume is

$$V = x(8 - 2x)(15 - 2x) = 4x^3 - 46x^2 + 120x.$$

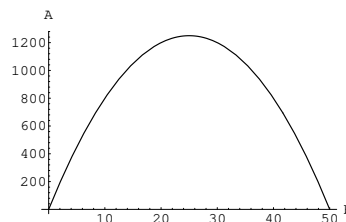
Therefore $V' = 12x^2 - 92x + 120 = 4(x - 6)(3x - 5)$, so $V' = 0$ is $x = 6$ or $x = 5/3$. We can ignore $x = 6$, because the largest square we can cut from each corner is 4 inches (because the cardboard is only 8 inches wide). Therefore $x = 5/3$; $V''(5/3) < 0$, so the box of maximum volume is $4\frac{2}{3}$ in $\times 11\frac{2}{3}$ in $\times 1\frac{2}{3}$ in.



According to the graph of V , the function does have a maximum at ≈ 1.67 .

3. Suppose the field runs b feet along the river and w feet perpendicular to the river. We want to maximize the area $A = bw$. The fencing must cover two sides of length w but only one of length b , so $L = b + 2w$. Therefore $b = L - 2w$, and we can express A solely in terms of w : $A = bw = w(L - 2w)$. This is an “upside-down” parabola whose roots are $w = 0$ and $w = L/2$. Therefore its vertex (and maximum) occurs halfway between—at $w = L/4$. (Alternately, use $A' = L - 4w$ and $A'' = -4 < 0$.) Hence $b = L/2$, so the maximum area is achieved by a rectangle that is twice as long as it is wide (and the long side runs along the river).

If $L = 100$ feet, then $A = w(100 - 2w)$. The graph of A , below, is a parabola whose vertex is at $w = 25 = 100/4$.



4. We want to maximize the volume of the storage bin, which is $V = x^2h$. The bin has a bottom whose area is x^2 and four sides whose areas are each xh . We are told that the total area $A = x^2 + 4xh$ is fixed. This implies $h = (A - x^2)/4x$, so we can express V solely in terms of x , as

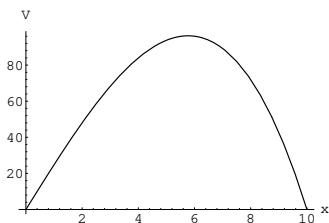
$$V(x) = x^2 \frac{A - x^2}{4x} = \frac{1}{4}(Ax - x^3).$$

Thus $V'(x) = \frac{1}{4}(A - 3x^2)$, and $V' = 0$ when $x = \sqrt{A/3}$. This is the dimension of the base, and $V''(\sqrt{A/3}) < 0$, so this gives maximum volume. The height is

$$h = \frac{A - A/3}{4\sqrt{A/3}} = 2 \frac{A/3}{\sqrt{A/3}} = 2\sqrt{A/3} = 2x.$$

Thus the height is twice the base.

If $A = 100$, then $V(x) = \frac{1}{4}(100x - x^3)$. The maximum does appear to be at $x = \sqrt{100/3} \approx 5.77$.



5. Suppose the radius of the semicircle is r . It sits on a rectangle whose dimensions are $2r$ and h . The length of the semicircle is πr , so the perimeter of the window is

$$L = \pi r + 2h + 2r.$$

The area of the semicircle is $\pi r^2/2$, and the area of the rectangle is $2rh$, so the area of the window is

$$A = \frac{\pi r^2}{2} + 2rh = \frac{\pi r^2}{2} + r(L - \pi r - 2r) = rL - \frac{4 + \pi}{2}r^2.$$

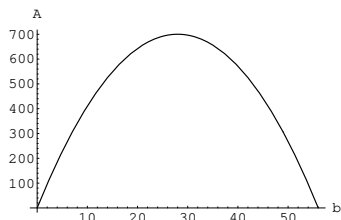
We've used the fact that the perimeter L is fixed to express A solely in terms of r . We've also done a bit of algebra to simplify the formula for A .

Now $A' = L - (4 + \pi)r$, so $A' = 0$ if $r = L/(4 + \pi)$, and $A'' = -(4 + \pi) < 0$. This determines the radius. To determine the height of the rectangle, we use the fact that $h = \frac{1}{2}(L - (\pi + 2)r)$:

$$\begin{aligned} h &= \frac{1}{2} \left(L - (\pi + 2) \frac{L}{4 + \pi} \right) = \frac{L}{2} \left(1 - \frac{\pi + 2}{4 + \pi} \right) \\ &= \frac{L}{2} \cdot \frac{4 + \pi - (\pi + 2)}{4 + \pi} = \frac{L}{4 + \pi} = r. \end{aligned}$$

Thus the height should equal the radius, implying that the rectangle is twice as wide as it is high.

When $L = 100$ then $A = 100r - (4 + \pi)r^2/2$. The graph is an upside-down parabola whose vertex (and maximum) is at $r = 100/(4 + \pi) \approx 14.0025$. In the figure below, $b = 2r$.



6. Our aim is to maximize the light transmitted. Suppose the clear glass transmits 1 unit of light per square foot; then the colored glass transmits 1/2 unit per square foot.

We use all pertinent information from the last question. The area of the semicircle is $\pi r^2/2$, so the transmitted light is $T = \pi r^2/4$. The area of the rectangle is $2rh$, so the transmitted light is $T = 2rh$; the total light transmitted is

$$T = \frac{\pi r^2}{4} + 2rh = \frac{\pi r^2}{4} + r(L - \pi r - 2r) = rL - \frac{8 + 3\pi}{4}r^2.$$

Now $T' = L - (8 + 3\pi)r/2 = 0$ when $r = 2L/(8 + 3\pi)$, and $T'' < 0$. This is less than the value of r found in the previous problem. In fact,

$$r_{\text{new}} = \frac{2L}{8 + 3\pi} < \frac{2L}{8 + 2\pi} = \frac{L}{4 + \pi} = r_{\text{old}}.$$

The new height is

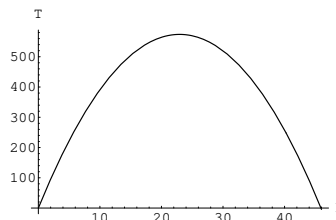
$$h = \frac{1}{2} \left(L - (\pi + 2) \frac{2L}{8 + 3\pi} \right) = \frac{L}{2} \frac{4 + \pi}{8 + 3\pi}.$$

This is larger than the radius; in fact,

$$h_{\text{new}} = \frac{4 + \pi}{4} \frac{2L}{8 + 3\pi} = \frac{4 + \pi}{4} r \approx 1.7854 r_{\text{new}}.$$

In comparison to the window in the last exercise, h is larger and r is smaller, so this window is taller and narrower than the previous one.

When $L = 100$, $T = 100r - \frac{8 + 3\pi}{4}r^2$ and T achieves its maximum when $r = 200/(8 + 3\pi) \approx 11.4779$. Once again, in the figure below $b = 2r$.



7. a. $C = 3(2\pi r^2) + 1(2\pi rh) + \frac{1}{2}(2\pi r) = 6\pi r^2 + 2\pi rh + \pi r$.

7. b. The volume is fixed at 100 cubic inches, so $\pi r^2 h = 100$. Therefore, $h = 100/(\pi r^2)$, so we can express C as a function of r alone:

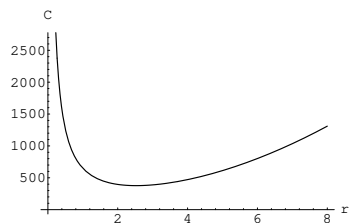
$$C(r) = 6\pi r^2 + \pi r + \frac{200\pi}{r}.$$

7. c. We want to choose r so that cost will be a minimum. Now

$$C'(r) = 12\pi r + \pi - 200\pi/r^2 = 0 \text{ if}$$

$$12r^3 + r^2 - 200 = 0.$$

By plotting $y = 12r^3 + r^2 - 200$ and magnifying, we find a root near $r = 2.527$ inches. The corresponding height is $h = 4.985$ inches.



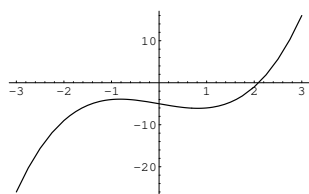
5.5 Newton's Method

1. Applying Newton's method to $x^2 - a = 0$, we get

$$x_{n+1} = x_n - \frac{(x_n^2 - a)}{2x_n} = \frac{1}{2} \left(x_n + \frac{a}{x_n} \right),$$

and the last expression is exactly what is used in the Babylonian algorithm.

2. The graph shows that this cubic has only one root. Newton's Method gives the root as $x = 2.094551481542327\dots$



3. Newton's method with $x_0 = 1$ gives the root as $x = 1.36880810\dots$. This happens in 4 steps.

4. $x = 1.1038034027\dots$

5. $x = .33473414\dots$ and $x = 1.21464804\dots$

6. a. We take the original function to be $f(x) = (1/x) - 3.4567$. Therefore $f'(x) = -1/x^2$ so the recursive step has the form

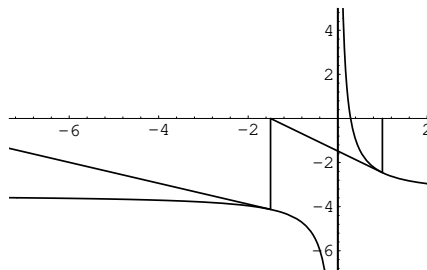
$$\begin{aligned} x_{n+1} &= x_n - \frac{1/x_n - 3.4567}{-1/x_n^2} = x_n + (x_n - 3.4567x_n^2) \\ &= 2x_n - 3.4567x_n^2. \end{aligned}$$

6. b. $x = .28929324\dots$

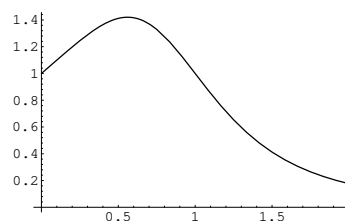
6. c. When $x_0 = 1$, iterates x_n "blow up" to $-\infty$:

n	x_n
0	1.0
1	-1.5
2	-10.2
3	-383.6
4	-509298.4
5	-896616684334.2

The graph of $\frac{1}{x} - 3.4567$ below shows what happens. The tangent at $x = 1$ angles up and meets the x -axis at $x = -1.4567 \approx -1.5$. The tangent at *this* point just continues to the left, so successive x_n move away from the root and towards $-\infty$.



7. a.

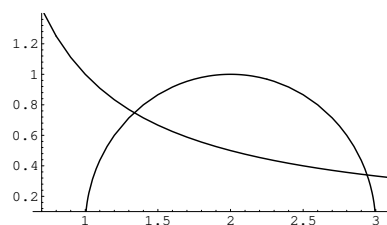


7. b. $f'(x) = (-3x^4 - 4x^3 + 1)/(x^4 + 1)^2$

7. c. $3x^4 + 4x^3 - 1 = 0$. $x = .56042566\dots$

7. d. $f(.56042566) \approx 1.42031925$.

8. a.



[Note: the axes origin has been shifted to $(.7, .1)$.]

8. b. $x^4 - 4x^3 + 3x^2 + 1 = 0$.

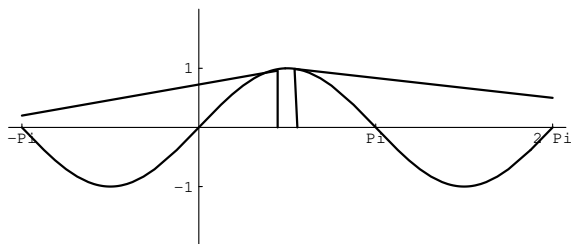
8. c. $x = 1.33653227\dots$ and $x = 2.94039266\dots$

9. a. If $x_0 = 1.55$, $x_1 \approx -46.5284$. If $x_0 = 1.56$, $x_1 \approx -91.0600$. If $x_0 = 1.57$, $x_1 \approx -1254.2831$. If $x_0 = 1.58$, $x_1 \approx 110.2287$.

9. b. The starting values x_0 are all near $\pi/2 \approx 1.5708$. For $g(x) = \sin(x)$, the basic formula takes the form $x_{n+1} = x_n - \tan(x_n)$. Since $\tan(x) \rightarrow +\infty$ as $x \rightarrow \pi/2$ from below while $\tan(x) \rightarrow -\infty$ as $x \rightarrow \pi/2$ from above, we see

$$x_1 = x_0 - \tan(x_0) = \begin{cases} \text{large negative} & x_0 < \pi/2 \\ \text{large positive} & x_0 > \pi/2. \end{cases}$$

By looking at tangents to the graph of $y = \sin(x)$ near $x = \pi/2$, we see what is happening even more clearly. The sine function has a maximum at $x = \pi/2$, so tangents from points close to $\pi/2$ are nearly horizontal. In particular, they meet the x -axis far from $x = \pi/2$ (and only accidentally near some root of the sine function!).



The Epidemic Runs Its Course

10. We have $(I + S - (b/a) \cdot \ln S)' =$

$$\begin{aligned} &= I' + S' - \frac{b}{a} \cdot \frac{S'}{S} \\ &= aSI - bI - aSI - \frac{b}{a} \cdot \frac{-aSI}{S} \\ &= -bI + bI \\ &= 0, \quad \text{as desired.} \end{aligned}$$

11. The function $I(t) + S(t) - (b/a) \cdot \ln S(t)$ is a constant because its derivative is zero, according to the last exercise.

12. From the graphs it appears that $I_\infty = 0$ for all $S(0)$.

13. Since $I(t) + S(t) - (b/a) \cdot \ln S(t) = C$, a constant, the limit of this expression as $t \rightarrow \infty$ exists and also has the value C . To determine C , we can simply evaluate the expression from a single t , say $t = 0$. Finally, $I_\infty = 0$ (from the previous exercise), so

$$\begin{aligned} S_\infty - \frac{b}{a} \cdot \ln(S_\infty) &= I_\infty + S_\infty - \frac{b}{a} \cdot \ln(S_\infty) \\ &= \lim_{t \rightarrow \infty} I(t) + S(t) - \frac{b}{a} \cdot \ln S(t) \\ &= C \\ &= I(0) + S(0) - \frac{b}{a} \cdot \ln S(0) \\ &= I_0 + S_0 - \frac{b}{a} \cdot \ln(S_0). \end{aligned}$$

14. $S_\infty = 460.93$.

15. a. $x - 8000 \ln(x) = -40615$. $S_\infty = 163.58$.

15. b. $x - 8000 \ln(x) = -55913$. $S_\infty = 1271.67$.

15. c. $x - 8000 \ln(x) = -63038$. $S_\infty = 4840.98$.

5.6 Chapter Summary

1. $T' = 500(2.20 - 2p) = 0$ if $p = 1.10$; $T'' < 0$. The best price is $p = 1.10$ dollars.

2. If $c = .30$ then $T' = 500(2.30 - 2p) = 0$ when $p = 1.15$. The best price is now $p = 1.15$ dollars. When c rises by .10, p rises by .05.

If $c = .50$ then $T' = 500(2.50 - 2p) = 0$ when $p = 1.25$. The best price is now $p = 1.25$ dollars. When c rises by .30, p rises by .15.

In each case, half of the rise in manufacturing cost is passed on to the customer.

3. $T' = 500(4.30 - 2p) = 0$ when $p = 2.15$. The best price is $p = 2.15$ dollars when $c = .30$ dollars.

If the unit manufacturing cost rises to .50, then $T' = 500(4.50 - 2p) = 0$ when $p = 2.25$. Thus, when c rises by .20, p rises by .10, or half as much.

4. If $c = .30$, then $T' = 100(15.30 - 2p) = 0$ when $p = 7.65$. If c rises to .50, then $T' = 100(15.50 - 2p) = 0$ when $p = 7.75$. Thus, when c rises by .20, p rises (once again) by .10, or half as much.

5. a. Here $T = g(p) = (a - mp)(p - c) = -mp^2 + (cm + a)p - ac$. Thus

$$T' = -2mp + cm + a = 0 \text{ when } p = \frac{c}{2} + \frac{a}{2m},$$

giving us the best price. In this formula it is clear that half of any change in c is passed on to p . This is the theoretical foundation for all the prior evidence.

5. b. Here $T = 1000 \frac{p - c}{1 + p^2}$, so

$$T' = 1000 \frac{1 + p^2 - 2p(p - c)}{(1 + p^2)^2} = 1000 \frac{1 + 2cp - p^2}{(1 + p^2)^2},$$

and $T' = 0$ when $1 + 2cp - p^2 = 0$. Solving this with the quadratic formula, we find

$$p = c + \sqrt{c^2 + 1}.$$

i. When $c = .50$, $p = .50 + \sqrt{1.25} = 1.62$.

ii. When $c = 1.00$, $p = 1.00 + \sqrt{2.00} = 2.41$.

iii. When $c = 1.50$, $p = 1.50 + \sqrt{3.25} = 3.30$.

Something remarkable happens here. When c increases by .50 (from .50 to 1.00), p increases by .79; when c increases by .50 again (from 1.00 to 1.50), p increases by .89. This means that *all* of the increase in c is passed on to the consumer, *and then some*. In fact,

$$\frac{dp}{dc} = 1 + \frac{c}{\sqrt{c^2 + 1}} \rightarrow 1 + 1 = 2$$

as $c \rightarrow \infty$. In other words, when costs get large, prices increase by *double* the amount that costs do!

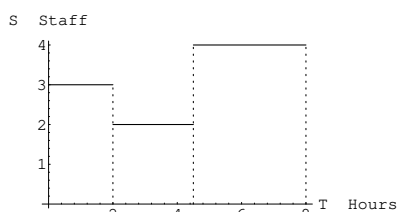
Solutions: Chapter 6

The Integral

6.1 Measuring Work

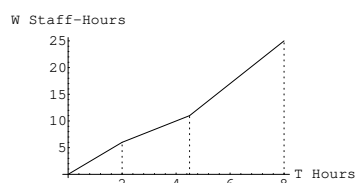
Human Work

1. a.



$$S(T) = (3 \times 2) + (2 \times 2.5) + (4 \times 3.5) = 25 \text{ staff-hours.}$$

1. b.

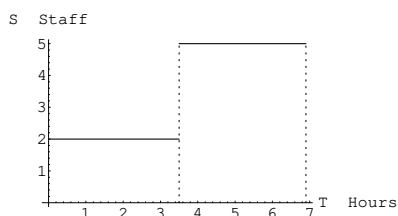


$$1. c. W'(T) = \begin{cases} 3 & 0 < T < 2, \\ 2 & 2 < T < 4.5, \\ 4 & 4.5 < T < 8. \end{cases}$$

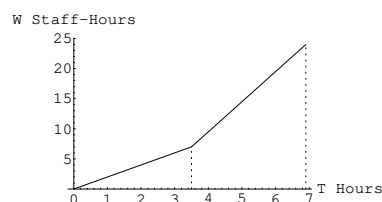
It is true that $W'(T) = S(T)$.

2. a. The job requires $4 \text{ staff} \times 6 \text{ hours} = 24 \text{ staff-hours}$. During the first 3.5 hours, $2 \times 3.5 = 7 \text{ staff-hours}$ are done, leaving $24 - 7 = 17 \text{ staff-hours}$ to be done. Since 5 staff are then available, the remaining job will take $17/5 = 3.4$ hours. The whole job therefore takes $3.4 + 3.5 = 6.9$ hours.

2. b.



2. c.

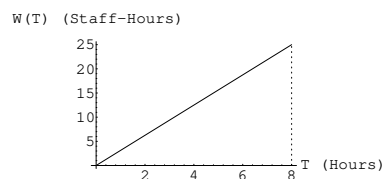


$$2. d. W'(T) = \begin{cases} 2 \text{ staff} & 0 < T < 3.5, \\ 5 \text{ staff} & 3.5 < T < 6.9. \end{cases}$$

Yes, $W'(T) = S(T)$.

3. Exercise 1: average is $25 \text{ staff-hours} / 8 \text{ hours} = 3\frac{1}{8}$ staff. Exercise 2: average is $24 \text{ staff-hours} / 6.9 \text{ hours} = 3.48$ staff.

4. a.



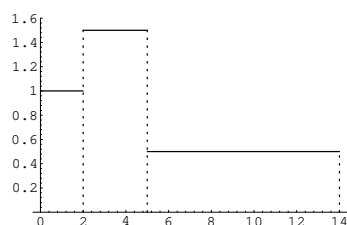
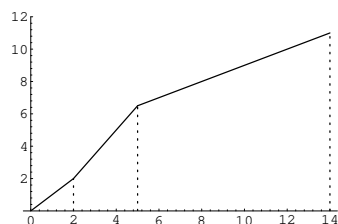
4. b. $W'(T) = 25 \text{ staff-hours} / 8 \text{ hours} = 3.125$ staff. $W'(T) = S(T) = 3.125$.

5. Average staffing = $42.5 \text{ staff-hours} / 11 \text{ hours} \approx 3.864$ staff.

Electrical Energy

6. $E = 1500 \times (11 - 7) = 6000 \text{ watt-hours}$ or 6 kilowatt-hours.

7. a. $E = 1000 \times (8 - 6) + 1500 \times (11 - 8) + 500 \times (20 - 11) = 11000 \text{ watt-hours}$ or 11 kwh.

7. b. Power demand $p(T)$:Accumulated energy consumption $E(T)$:

The slope at any point on the second graph equals the height at the corresponding point on the first graph, so once again $E'(T) = p(T)$.b

7. c. Average power demand = 11 kwh / 14 hr $\approx$.7857 kw. If the heater could be set at the average power level of $\approx$ 785.7 watts, the energy consumption would still be 11 kwh.

Work as Force $\times$ Distance

8. Average power demand = $\frac{1447 \text{ mwh}}{24 \text{ hours}} \approx 60.29 \text{ mw}$.

9. $W(T) = (150 \times 200) + (400 \times 550) = 250,000$ foot-pounds.

10. a. Weight = $2000 + 40 \times (30 - x) = (3200 - 40x)$ pounds.

10. b. $W_1 = 10 \times (2000 + (40 \times 30)) = 32000$ foot-pounds.

$W_2 = 10 \times (2000 + 40 \times 20) = 28000$ foot-pounds.

$W_3 = 10 \times (2000 + 40 \times 20) = 24000$ foot-pounds.

Total work = $32000 + 28000 + 24000 = 84000$ foot-pounds.

10. c. There are now 30 terms; each of the form

$$W = 1 \text{ ft} \times (2000 + 40L) \text{ lbs} = 2000 + 40L \text{ ft-lbs.}$$

At the start $L = 30$, it drops by 1 each time and ends at 1. Thus

$$\begin{aligned} W &= (2000 + 40 \cdot 30) + (2000 + 40 \cdot 29) + \cdots \\ &\quad + (2000 + 40 \cdot 1) \\ &= 30 \cdot 2000 + 40(30 + 29 + \cdots + 1) \end{aligned}$$

To see that $1 + 2 + \cdots + 30 = 465$ you can either write a little computer program or use the formula $1 + 2 + \cdots + n = n(n+1)/2$. Therefore,

$$W = 30 \cdot 2000 + 40 \cdot 465 = 78600 \text{ ft-lbs.}$$

This estimate is much smaller than the one in part (b). The inaccuracy in each estimate is due to assuming that the weight of the load is unchanged during the 1 foot or 10 feet over which an individual calculation is made. The assumption is likely to cause a greater error for the 10-foot calculation, so the estimate here in part (c) is likely to be more accurate.

10. d. The work involved in raising only the anchor is $W = 2000 \times 30 = 60000$ foot-pounds. The extra work in raising the chain is therefore $78600 - 60000 = 18600$ foot-pounds.

6.2 Riemann Sums

Making Approximations

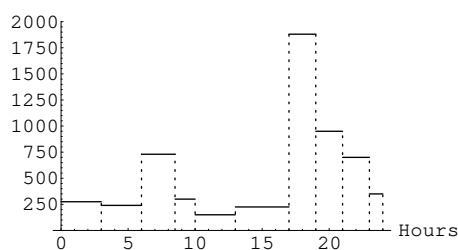
1. Since the ship travels 61.65 miles in 15 hours, its average velocity is $(61.65 \text{ miles})/(15 \text{ hours}) \approx 4.11 \text{ mph}$.

2. a. *Ad libitum*. There is a wide latitude of choices for the sub-intervals, Δt_k , so answers will vary. However, the sub-intervals Δt_k must add up to 24. Also, the individual times 1:30, 5:00, 8:00 must lie in successive Δt_k .

Here is one possibility: Energy $\approx (275 \times 3) + (240 \times 3) + (730 \times 2.5) + (300 \times 1.5) + (150 \times 3) + (225 \times 4) + (1880 \times 2) + (950 \times 2) + (700 \times 2) + (350 \times 1)$. (Note that the sum of the time intervals is indeed 24.

2. b. Using the above choice of Δt_k , we estimate the energy consumption to be 12580 watt-hours. The results in parts (c) and (d) reflect the same choice.

2. c. Power (watts)



2. d. Average power is

$$\frac{\text{energy used}}{\text{time elapsed}} = \frac{12580 \text{ watt-hours}}{24 \text{ hours}} \approx 524.167 \text{ watts.}$$

3. Each term has the form $.1P(t) \times 1 = .1P(t)$:

$$\begin{aligned} W &= .1P(.5) + .1P(1.5) + .1P(2.5) + .1P(3.5) \\ &= 30(e^{0.1} + e^{0.3} + e^{0.5} + e^{0.7}) \\ &\approx 30(1.11 + 1.35 + 1.65 + 2.01) \\ &= 183.6 \text{ grams.} \end{aligned}$$

Using RIEMANN

4. a. The values have stabilized to two decimal places.

n	sum
40	6.133337...
400	6.220269...
4000	6.228990...
40000	6.229862...

4. b. To 4 decimal places the limiting value is 6.2299.

4. c. 45.8197

4. d. 52.0496. The numerical value of the limit on interval $[1, 7]$ is the sum of the numerical value of the limits on the intervals $[1, 3]$ and $[3, 7]$.

5. Change $x = a$ to $x = a + \text{deltax} / 2$.

6. a. Seven decimal places have stabilized.

n	sum
40	6.22980412...
400	6.22995783...
4000	6.22995937...
40000	6.22995938...

6. b. On the interval $[1, 3]$, roughly 18 subintervals are needed to stabilize the first four digits of the midpoint Riemann sum. The stable value is $\sqrt{1+x^3} = 6.229\dots$. Though left endpoint Riemann sums require more subintervals to stabilize to four digits, they yield the same limit value as the midpoint sums.

6. c. For this function midpoint Riemann sums are more “efficient” by several orders of magnitude.

7. a. To get *right endpoint* Riemann sums, replace $x = a$ by $x = a + \text{deltax}$. As was the case with left endpoint Riemann sums, four digits stabilize.

n	sum
40	6.3272...
400	6.2396...
4000	6.2309...
40000	6.2300...

7. b. Right and left endpoint sums are about equally efficient; however, midpoint Riemann sums are the most efficient.

8. The following estimates agree with the values given on page 354.

n	sum
20	1.552259...
50	1.566098...

9. a. The following estimates agree with the values given on page 356.

n	sum
4	3.819940...
20	3.820197...

9. b. The limit to 11 decimal places is 3.820 197 789 02.

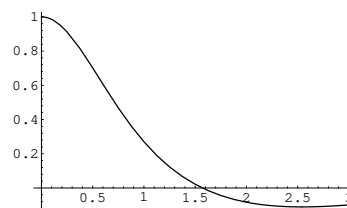
- 10.

n	sum
100	.63392097...
1000	.59837871...
10000	.59485189...

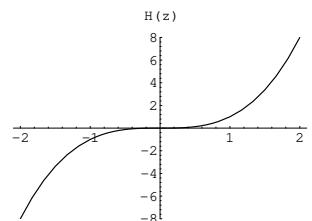
- 11.

n	sum
10	-.1032077...
100	-.1040498...
1000	-.1041221...

On the interval $[2, 3]$, $f(x) = \cos x / (1 + x^2)$ is entirely negative. (See the graph below.) Thus, every term in a Riemann sum for f on this interval will have a negative factor. The sum itself must therefore be negative.



12. a.



Note that the graph of $H(z) = z^3$ is rotationally symmetric around the origin. When a Riemann sum contains an even number of terms (e.g., 10, 100, or 1000, as in this exercise), and midpoints are sampled, then every sample to the *left* of the origin (at a point z) corresponds exactly to a sample to the *right* of the origin (at $-z$). The sample values are therefore negatives of each other (namely

z^3 and $(-z)^3 = -z^3$, so the terms they contribute to the Riemann sum exactly cancel. Thus the whole Riemann sum is 0.

12. b. The left endpoint Riemann sums are all negative when there are an even number of terms. Here is why. The value of $H(z)$ increases as $|z|$ increases. Consider two rectangles symmetrically placed about the origin (as in the previous part). The height of each rectangle is determined by the value of $H(z)$ at its left end. For the rectangle to the left of the origin, its left end is *farther* from the origin than its right end; for the rectangle to the right of the origin, its left end is *closer* to the origin than its right end. Therefore, the rectangle to the left of the origin is taller than the corresponding one to the right. When the two are added together, the contribution from the one on the left dominates. Since $H(z)$ is negative for each of those rectangles, the negative terms dominate, leading to the whole Riemann sum being negative.

n	sum
10	-3.2000...
100	$-3.1999... \times 10^{-1}$
1000	$-3.1999... \times 10^{-2}$

Volume as a Riemann Sum

13. Estimate the volume of an individual slice is its thickness (2 cm) times its area (which varies). Then estimate the total volume to be the sum:

$$\begin{aligned} V &\approx 2(72 + 145 + 139 + 127 + 111 + 89 + 63 + 22) \\ &= 2 \times 768 = 1536 \text{ cm}^3. \end{aligned}$$

14. a. $V = 4\pi \times 3^3/3 \approx 113.0973.$

14. b.

n	Riemann sum
100	113.10298...
1000	113.09739...
10000	113.09733...

This value agrees with the value given in part (a).

15. a. The shape is a circular cone whose base has radius 3 and whose altitude is 6. The vertex of the cone is at the origin and its axis lies along the positive x -axis. If r is the radius and h the altitude, then the volume is

$$V = \frac{1}{3}\pi r^2 h = 18\pi \approx 56.5487.$$

15. b. The following are midpoint Riemann sums for $A(x) = \pi(r(x))^2$, where $r(x) = .5x$. The sum stabilizes

to four decimal places; when it is rounded, the sum agrees with the theoretical value from part (a).

n	sum
100	56.54725...
1000	56.54865...
10000	56.54866...

Summation Notation

16. a. $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55.$

16. b. $1 + 4 + 9 + 16 + 25 = 55.$

16. c. $1 + 3 + 5 + 7 + 9 = 25.$

17. $\sum_{k=1}^5 2k.$

18. a. The sum, when written out, is

$$1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \frac{1}{5} - \frac{1}{6}$$

All the intermediate terms cancel, leaving only the first and last: $1 - \frac{1}{6} = \frac{5}{6}.$

18. b. This is like the last: all the intermediate terms will cancel, leaving only the first and last: $1 - \frac{1}{501} = \frac{500}{501}.$

19. a. $\sum_{k=1}^n k^2.$ **19. c.** $\sum_{k=1}^{12} f(s_k) \Delta s.$

19. b. $\sum_{k=1}^m 2^k.$ **19. d.** $\sum_{k=1}^n y_k^2 \Delta y_k.$

20. a. $a_3 + a_4 + a_5 + \cdots + a_{n-2} + a_{n-1}.$

20. b. $\frac{1}{1} + \frac{2}{2} + \frac{3}{5} + \frac{4}{10} + \frac{5}{17}.$

20. c. $H(x_1) \Delta x_1 + H(x_2) \Delta x_2 + H(x_3) \Delta x_3 + H(x_4) \Delta x_4 + H(x_5) \Delta x_5.$

21.

n	$\sum_{k=1}^n k$	$\left(\sum_{k=1}^n k\right)^2$	$\sum_{k=1}^n k^3$
2	3	9	9
3	6	36	36
4	10	100	100
5	15	225	225
6	21	441	441

22. $\Delta u = .5$; the endpoints are 0, .5, 1, and 1.5; the function values are $g(0) = 25$, $g(.5) = 25 - .25 = 24.75$,

$g(1) = 25 - 1 = 24$, and $g(1.5) = 25 - 2.25 = 22.75$.
Therefore,

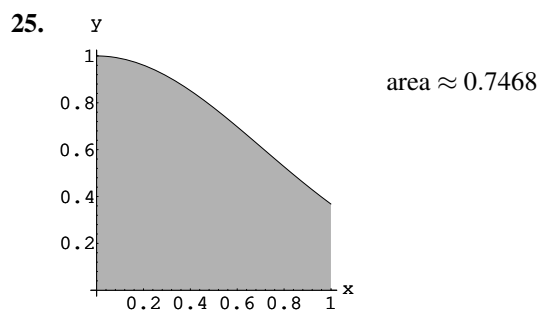
$$\begin{aligned}\text{sum} &= .5g(0) + .5g(.5) + .5g(1) + .5g(1.5) \\ &= .5 \times (25 + 24.75 + 24 + 22.75) \\ &= 48.25\end{aligned}$$

Length and Area

23. With midpoint Riemann sums, the value stabilizes to four decimal places with around 300 subintervals. With endpoint Riemann sums, more subintervals are needed.

n	midpoint sum
200	1.478940
300	1.478942
400	1.478942
500	1.478943

24. a. .3333... **b.** 8.9999... **c.** 3.1415...



26. a. The length of the graph of $y = \ln x$ over the interval $[1, e]$ is 2.0034.... Riemann sums are calculated using the length function $y = \sqrt{1 + 1/x^2}$.

26. b. The length of the graph of $y = \exp(x)$ over the interval $[0, 1]$ is also 2.0034.... This time the length function is $y = \sqrt{1 + \exp^2(x)}$.

27. a. The length of the hyperbola $y = 1/x$ over the interval $[1, 4]$ is 3.1501.... The length function is $y = \sqrt{1 + 1/x^4}$.

27. b. The area under the hyperbola over the same interval is 1.3862....

28. a. The length function is

$$y = \sqrt{1 + \frac{x^2}{4 - x^2}} = \sqrt{\frac{4}{4 - x^2}}.$$

It takes roughly 1,000,000 subintervals for left endpoint Riemann sums to stabilize to 3.14159....

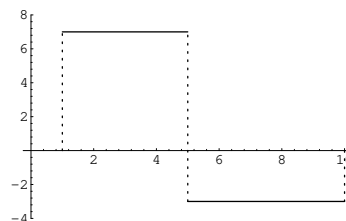
28. b. If right endpoints are used, then we eventually need to sample the length function from part (a) at the point $x = 2$. This yields an infinite value, so the length function will be undefined at the right endpoint of interval $[0, 2]$.

6.3 The Integral

1. a. 39 **1. c.** 14 **1. e.** -26

1. b. -39 **1. d.** -14 **1. f.** +26

2. a.



2. b. $\int_1^7 g(x) dx = 22$; $\int_7^{10} g(x) dx = -9$;
 $\int_1^{10} g(x) dx = 13$

3. a. $\int_0^1 \frac{1}{1+x^3} dx = .8356$.

3. b. Left endpoint sums require 100,000 subintervals to get four decimal places accuracy, while midpoint sums require only 1,000 subintervals.

4. a. Students should show tables used to construct their estimates.

$\int_1^4 \sqrt{1+x^3} dx = 12.8714...$

4. b. $\int_4^7 \sqrt{1+x^3} dx = 39.1785...$

4. c. $\int_0^3 \frac{\cos x}{1+x^2} dx = .6244...$

5. a. $\int_1^2 e^{-x^2} dx = 0.1352...$

5. b. $\int_0^4 \cos(x^2) dx = 0.5944...$

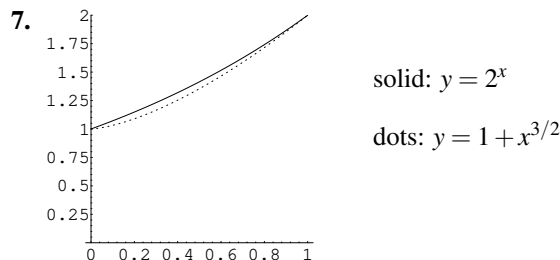
5. c. $\int_0^4 \sin(x^2) dx = 0.7471...$

5. d. $\int_0^1 \frac{4}{1+x^2} dx = 3.1415...$

6. a. 3.1678...

6. b. 3.1678...

6. c. The lengths are the same because the curve $y = \sqrt{x}$ (from $x = 1$ to $x = 4$) is just the reflection of the curve $y = x^2$ (from $x = 1$ to $x = 2$) across the line $y = x$.



The length of $y = 2^x$ (the solid curve) is 1.421162; the length of $y = 1 + x^{3/2}$ is 1.439710. Therefore, the solid curve is slightly shorter. Even though the lengths are nearly the same, it is obvious to the eye that the solid curve is shorter.

8. The area of the base is $2 \times 30^2 = 1800$ square feet. The volume of the pyramid is

$$\int_0^{30} 2x^2 dx = 18,000 \text{ cubic feet.}$$

The integral is estimated by midpoint Riemann sums.

9. a. The true value of the integral is *smaller* than the estimate. The function is positive and decreasing on the interval $[1, 2]$, so left endpoint rectangles lie *above* the graph. Their total area is therefore larger than the area under the graph.

9. b. The error bound is

$$\Delta x |f(2) - f(1)| = .001(e^{-1} - e^{-4}) < .000350.$$

9. c. $A = .135432 - .000350 = .135082$, $B = .135432$.

9. d. The first three digits are known: .135.

10. a. The true value of this integral is *larger* than the estimate. The function is positive and increasing on the interval $[0, \pi/4]$, so left endpoint rectangles lie *below* the graph. Their total area is therefore smaller than the area under the graph.

10. b. The error bound is $\Delta x |f(\pi/4) - f(0)| = (\pi/400) \tan(\pi/4) = .007854$. ($\tan \pi/4 = 1$.)

10. c. $A = 0.342652$, $B = .342652 + .007854 = .350506$.

10. d. Only the first digit is known: .3.

11. a. A left endpoint Riemann sum with 100 equal subintervals gives

$$\int_0^{\pi/2} \sin x dx = .9921 \dots$$

The true value of the integral is larger than this because $\sin x$ is positive and increasing on the interval $[0, \pi/2]$.

Left endpoint Riemann sums are always underestimates in these circumstances.

11. b. The error bound is $\Delta x |\sin \pi/2 - \sin 0| = \pi/200 < .0157$. Hence $A = .9921$, $B = .9921 + .0157 = 1.0078$.

12. a. The true value of the integral is larger than the left endpoint estimate because $\sqrt{1+x^3}$ is positive and increasing on the interval $[1, 3]$. Left endpoint Riemann sums are always underestimates in these circumstances.

12. b. The error bound is .007755.

12. c. $A = 6.226083$, $B = 6.226083 + .007755 = 6.233838$.

12. d. The first two digits are known: 6.2.

13. a. $\bar{f} = 5$.

13. b. $\bar{f} = 0$ on the intervals $[0, 2\pi]$ and $[0, 100\pi]$.

13. c. $\bar{f} = \frac{1}{2}$ on the intervals $[0, 2\pi]$ and $[0, 100\pi]$. This can be seen either by constructing estimates for

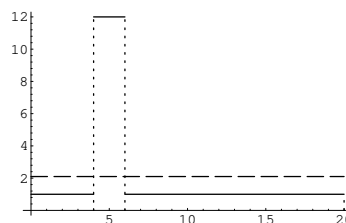
$$\frac{1}{2\pi} \int_0^{2\pi} \sin^2 x dx$$

or by noticing that the graph $y = \sin^2 x$ separates the rectangle $0 \leq x \leq 2\pi$, $0 \leq y \leq 1$ into two congruent regions: one below the graph and one above. Therefore the area *under* the graph must be half the area of the rectangle (which is 2π).

14. a. $\bar{H} = ((1 \times 4) + (12 \times 2) + (1 \times 14))/20 = 2.1$

14. b. $\bar{H}$ is smaller than the average of 12 and 1.

14. c.

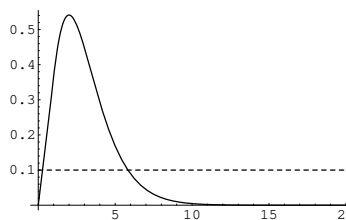


The area under $y = H(x)$ is equal to the area under the horizontal dashed line, $y = \bar{H}$.

15. a. The minimum is 0, the maximum is 0.544, and the average of the two is 0.272.

15. b. $\bar{f} = 0.1$.

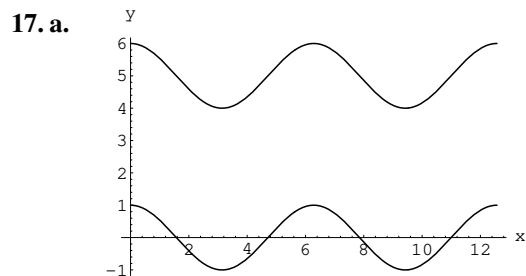
15. c.



The graph has a relatively narrow peak, so “most” values of $f(x)$ lie below the average of the maximum and the minimum (which has the value .272). Hence $\bar{f}$ will lie below that point as well.

The Integral as a Signed Area

16.	n	sum
	10	2.5
	100	2.5000...
	1000	2.4999...
	10000	2.4999...
	100000	2.5000...
	1000000	2.5000...



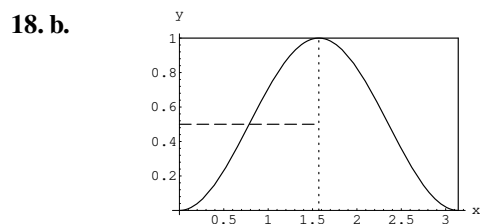
17. b. The area above the x -axis is equal to the area below the x -axis, therefore $\int_0^{4\pi} \cos x dx = 0$.

17. c. $\int_0^{4\pi} 5 + \cos x dx = 20\pi$. $\int_0^{4\pi} 5 dx$ has the same value because

$$\int_0^{4\pi} 5 + \cos x dx = \int_0^{4\pi} 5 dx + \int_0^{4\pi} \cos x dx,$$

and the second integral is 0, by part a.

18. a. $\int_0^{\pi} \sin^2 x dx = 1.5707 \dots$

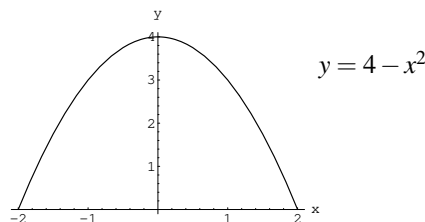


18. c. The area under the graph is exactly *half* the area of the rectangle (which is $\pi \times 1 = \pi$). To see why, look at $y(x)$ for $0 \leq x \leq \pi/2$. On this interval, the curve is symmetric about the point at which $y(x)$ intersects the horizontal line $y = 0.5$. Therefore on the interval $[0, \pi/2]$, the area above the curve is the same as the area below. The other half is the same.

18. d. Because the area under the curve of $\sin^2 x$ over the interval $[0, \pi]$ is half the area of the rectangle drawn above, $\int_0^{\pi} \sin^2 x dx$ is exactly $\pi/2 \approx 1.5707963$.

19. a. The graph of $y = 4 - x^2$ lies above the axis on the interval $-2 < x < 2$.

19. b.

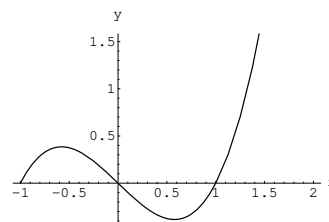


19. c. Midpoint Riemann sums determine that the area above the x -axis and below the graph is

$$\int_{-2}^2 4 - x^2 dx = 10.666 \dots$$

20. a. The signed area is $\int_{-1}^2 x^3 - x = 2.25 \dots$

20. b.

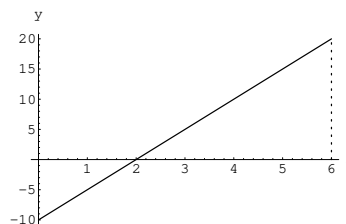


Notice that the (absolute, non-signed) areas over the intervals $[-1, 0]$ and $[0, 1]$ are equal but opposite in sign. Therefore, in the *signed* area, their contributions cancel. Thus the signed area over the interval $[-1, 2]$ reduces simply to the signed area over $[1, 2]$.

6.4 The Fundamental Theorem of Calculus

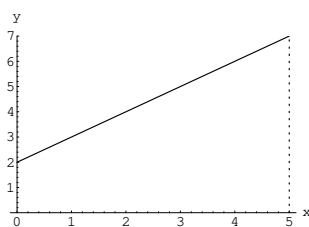
Constructing Accumulation Functions

1. a. $A(x) = 5x - 5(2) = 5x - 10$.

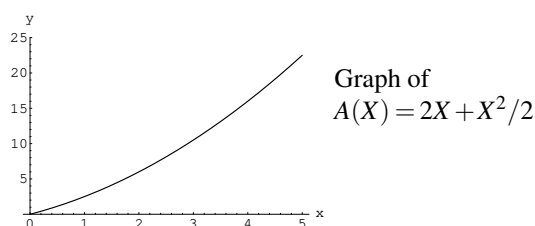


1. b. Yes, $A'(x) = 5$.

2. a.



2. b.



2. c. $A'(x) = 2 + x$.

2. d. The comparison simply has to be carried out.

3. a. $A(x) = \frac{1}{4}x^4$.

3. b. $A(x) = \frac{1}{4}x^4 - \frac{1}{4}$. The formulas for A in parts (a) and (b) differ by the constant $-\frac{1}{4}$ that has been added to the formula in part (b). It has been added because the initial condition for (b) is $A(1) = 0$, which is different from (a), where $A(0) = 0$.

Using the Fundamental Theorem

4. a. $A'(x) = \cos(x)$. 4. e. $A'(x) = \sin(x^2)$.

4. b. $A'(x) = \sin(x)$. 4. f. $A'(x) = \sin^2(x)$.

4. c. $A'(x) = \cos(x^2)$. 4. g. $A'(x) = \ln(x)$.

4. d. $A'(x) = \cos(x^2)$. 4. h. $A'(x) = x^2 - 4x^3$.

5. The critical points are the zeros of the function $A'(x) = \cos(x^2)$. The local maxima are located at $x = \sqrt{\pi/2}$, $\sqrt{5\pi/2}$, and $\sqrt{9\pi/2}$. The local minima are located at $x = \sqrt{3\pi/2}$ and $\sqrt{7\pi/2}$.

6. There are six critical points in the interval $[0, 4]$. The local minima are located at $x = 0$, $\sqrt{2\pi}$, and $\sqrt{4\pi}$. The local maxima are located at $x = \sqrt{\pi}$, $\sqrt{3\pi}$, and $\sqrt{5\pi}$.

7. There are two critical points, $x = 0$ and $x = 1/4$. They are the zeros of $A'(x) = x^2 - 4x^3 = x^2(1 - 4x)$. By noticing $A' > 0$ when $x < 1/4$ and $A' < 0$ when $x > 1/4$, we can see that A has a local maximum at $x = 1/4$.

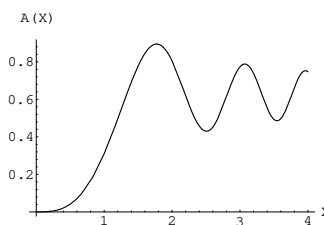
8. a. $y = \int_{\sqrt{\pi}}^x \cos(x^2) dx$.

8. b. $y = \int_0^x \sin(x^2) dx$.

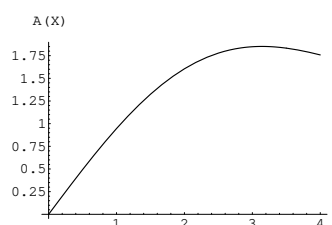
8. c. $y = 5 + \int_0^x \sin(x^2) dx$.

8. d. $y = \int_0^x e^{-x^2} dx$.

9. a.



9. b.



Formulas for Integrals

10. a. $474\frac{2}{3}$

10. b. 2

10. c. 0

10. d. $e - 1$

11. a. $20k$

11. b. $(1 - \cos(\alpha\pi))/\alpha$

11. c. $21p - 63\frac{3}{4}$

11. d. $(e^c - 1)/c$

12. a. $y(x) = \frac{1}{3}x^3 - x^4$

12. b. $y(x) = \frac{1}{3}x^3 - x^4 + 72$

12. c. $y(x) = \frac{1}{3}x^3 - x^4 + 82$

12. d. $y(x) = \sin(3x)/3$

13. a. $\frac{1}{12}$

b. $\frac{1}{e-1}$

c. $\frac{2}{\pi}$

14. a. $\frac{p}{2} - \frac{1}{3}$

b. $p = \frac{2}{3}$

10. e. $\ln 6$

10. f. -8136

10. g. $1/\ln 2 = 1.442695$

10. h. $2/3$

11. e. $(3^c - 2^c)/c$

11. f. $5b - \frac{1}{2}b^2 - 4\frac{1}{2}$

11. g. $(a-1)/\ln a$

11. h. $(2^{c+1} - 1)/(c+1)$

Solutions: Chapter 7

Periodicity

7.1 Periodic Behavior

There are no exercises in this section.

7.2 Period, Frequency, and the Circular Functions

Circular Functions

1. a. 1 b. $1/5$ c. $1/2\pi$ d. $1/\pi$ e. 3

2. a.

	$\sin(x)$	$\sin(2x)$	$\sin(x) + \sin(2x)$
per	2π	π	2π
freq	$1/2\pi$	$1/\pi$	$1/2\pi$

2. b.

	$\sin(2x)$	$\sin(3x)$	$\sin(2x) + \sin(3x)$
per	π	$2\pi/3$	2π
freq	$1/\pi$	$3/2\pi$	$1/2\pi$

2. c.

	$\sin(6x)$	$\sin(9x)$	$\sin(6x) + \sin(9x)$
per	$\pi/3$	$2\pi/9$	$2\pi/3$
freq	$3/\pi$	$9/2\pi$	$3/2\pi$

3. We can write

	$\sin(ax)$	$\sin(bx)$	$\sin(ax) + \sin(bx)$
per	$2\pi/a$	$2\pi/b$	$2\pi/Q$
freq	$a/2\pi$	$b/2\pi$	$Q/2\pi$

where Q depends on a and b . If a divides b (that is, b is an integer multiple of a , even if a and b aren't integers), then $Q = a$. Likewise, if b divides a , then $Q = b$. If neither divides the other, then Q is the least common multiple (**l.c.m.**) of a and b , divided by the product of a and b . Here is why. The sum will repeat only when a whole number of periods of the first term coincides with a whole number of periods of the second. Thus,

$$\frac{2\pi}{Q} = m \cdot \frac{2\pi}{a} = n \cdot \frac{2\pi}{b},$$

where we can assume that m and n are the *smallest* positive integers for which this equation is true. But then $abQ = mb = na$, and this number is the **l.c.m.** of a and b . Therefore, $Q = \text{l.c.m.}/ab$, as claimed.

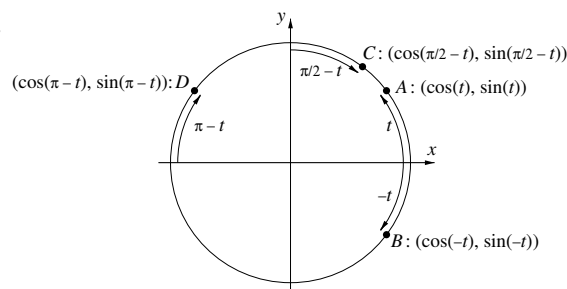
4. a. Amplitude: 5, frequency: $3/2\pi$.

4. b. Amplitude: 15, frequency: $3/2\pi$, because $g'(x) = -15\sin(3x)$.

5. a. $\int_0^x 5\cos(3t) dt = \frac{5}{3}\sin(3t) \Big|_0^x = \frac{5}{3}\sin(3x)$; this is periodic.

5. b. Amplitude: $5/3$, frequency: $3/2\pi$.

6.



The points A, B, C, and D are located at t , $-t$, $\pi/2 - t$, and $\pi - t$ radians, respectively. We have:

i. The x coordinates of B and A are the same but their y coordinates are negatives of each other:

$$\cos(-t) = \cos(t) \quad \sin(-t) = -\sin(t).$$

ii. The y coordinate of C is the x coordinate of A, and vice versa:

$$\sin\left(\frac{\pi}{2} - t\right) = \cos(t) \quad \cos\left(\frac{\pi}{2} - t\right) = \sin(t).$$

iii. The y coordinates of D and A are the same but their x coordinates are negatives of each other:

$$\sin(\pi - t) = \sin(t) \quad \cos(\pi - t) = -\cos(t).$$

The graphs of the sine and cosine functions reflect these properties in the following ways:

- $\sin(-t) = -\sin(t)$: Reflecting $y = \sin(t)$ across the y -axis produces the same result as reflecting it across the t -axis.
- $\cos(-t) = +\cos(t)$: Reflecting $y = \cos(t)$ across the y -axis produces the same result as doing nothing.
- $\sin(\pi/2 - t) = \cos(t)$: Reflecting $y = \sin(t)$ across the vertical line $t = \pi/4$ (not $t = \pi/2$!) produces the graph $y = \cos(t)$.
- $\sin(\pi - t) = \sin(t)$: Reflecting $y = \sin(t)$ across the vertical line $t = \pi/2$ (not $t = \pi$!) produces the same result as doing nothing.

$$7. \text{ a. } \frac{1}{\pi} \int_0^{\pi} \sin(s) ds = -\frac{1}{\pi} \cos(s) \Big|_0^{\pi} = \frac{2}{\pi}.$$

$$7. \text{ b. } \frac{1}{\pi} \int_{\pi/2}^{3\pi/2} \sin(s) ds = -\frac{1}{\pi} \cos(s) \Big|_{\pi/2}^{3\pi/2} = 0.$$

$$7. \text{ c. } \frac{1}{2\pi} \int_0^{2\pi} \sin(s) ds = -\frac{1}{2\pi} \cos(s) \Big|_0^{2\pi} = 0.$$

$$7. \text{ d. } \frac{1}{2\pi} \int_c^{c+2\pi} \sin(s) ds = -\frac{1}{2\pi} \cos(s) \Big|_c^{c+2\pi} = 0.$$

7. e. The average value of $\sin(s)$ over a full period is 0. Therefore, it does not depend on the point c where the period begins. As parts (a) and (b) show, the average value over a half period *does* depend on the starting point c .

$$8. \text{ a. } T = 2\pi/b.$$

8. b. The average value of $P(t)$ over $c \leq t \leq c + T$ is

$$\begin{aligned} \frac{1}{T} \int_c^{c+T} A \sin(bt) dt &= \frac{b}{2\pi} \left(-\frac{A}{b} \cos(bt) \right) \Big|_c^{c+2\pi/b} \\ &= -\frac{A}{2\pi} (\cos(bc + 2\pi) - \cos(bc)) = 0. \end{aligned}$$

The average value over a full period is always 0; in particular, the value does not depend on the choice of c .

8. c. The average value of $P(t)$ over $0 \leq t \leq T/2$ is

$$\begin{aligned} \frac{2}{T} \int_0^{T/2} A \sin(bt) dt &= \frac{b}{\pi} \left(-\frac{A}{b} \cos(bt) \right) \Big|_0^{\pi/b} \\ &= -\frac{A}{\pi} (\cos(\pi) - \cos(0)) = \frac{2A}{\pi}. \end{aligned}$$

9. The graph of $y = f(x - \phi)$ is just the graph of $y = f(x)$ shifted to the right by the amount ϕ . If we shift the graph of $y = \sin(x)$ to the right by $3\pi/2$, we will get the graph of $y = \cos(x)$. Therefore, $\phi = 3\pi/2$.

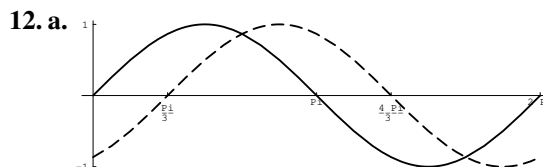
Because $\sin(x)$ is periodic with period 2π , a shift of -2π is equivalent to no shift at all. Therefore a shift

of $3\pi/2$ is equivalent to a shift of $-2\pi + 3\pi/2 = -\pi/2$. Consequently, we can also say $\phi = -\pi/2$ (or, more generally, $3\pi/2 + 2n\pi$, for any integer n).

Phase

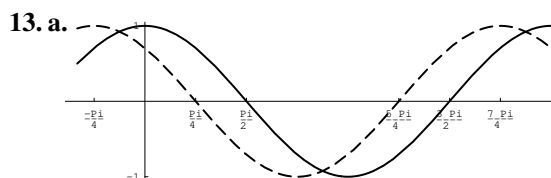
10. A shift of π converts the graph of $y = \sin(x)$ into the graph of $y = -\sin(x)$. Therefore, $\phi = \pi + 2n\pi$.

11. A shift of $\pi/2$ converts the graph of $y = \sin(x)$ into the graph of $y = -\cos(x)$. Therefore, $\phi = \pi/2 + 2n\pi$.



12. b. By definition, the phase difference between $\sin(t)$ and $\sin(t - \pi/3)$ is $\pi/3$.

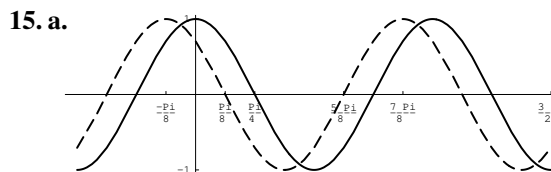
12. c. Since a shift of $\pi/3$ moves the graph of $y = \sin(t)$ onto the graph of $y = \sin(t - \pi/3)$, the phase shift is (also) $\pi/3$.



13. b. By definition, the phase difference is $\phi = -\pi/4$.

13. c. The phase shift is $\phi/b = -\pi/4$, because we get the graph of $y = \cos(t + \pi/4)$ when we shift the graph of $y = \cos(t)$ by $\pi/4$ units to the left (or $-\pi/4$ units to the right).

14. The point closest to the origin where $\cos(t + \pi/4)$ reaches its maximum value of 1 is at $t = -\pi/4$ (see the graph in 13a). Yes, this maximum is shifted from the first by exactly the phase shift in exercise 13. The graph demonstrates this, too.



The phase difference is $-\pi/4$, but the graph shows that the phase shift is $-\pi/8$. In particular, the maximum that $\cos(2t)$ has at the origin is shifted to the maximum that $\cos(2t + \pi/4)$ has at the point where $t = -\pi/8$.

16. We must show $y = A \sin(bt - \varphi) = 0$ when $t = \varphi/b$:

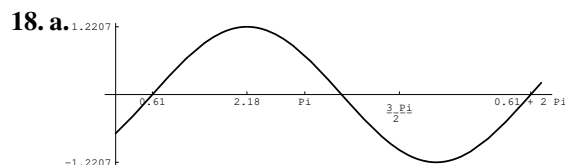
$$y = A \sin(b\varphi/b - \varphi) = A \sin(\varphi - \varphi) = A \sin(0) = 0.$$

17. a. The function $y = A \cos(bt - \varphi)$ achieves its maximum value (which is A) at $t = \varphi/b$. We can see this by finding where $y' = 0$:

$$y' = -bA \sin(bt - \varphi) = 0 \quad \text{if} \quad bt - \varphi = 0,$$

so $t = \varphi/b$.

17. b. We already know $y = \cos(bt - \varphi)$ is just $y = \cos(bt)$ shifted by *some* amount. Since $\cos(bt)$ has its maximum at $t = 0$ while $y = \cos(bt - \varphi)$ has its maximum at $t = \varphi/b$, the phase shift must be exactly φ/b .



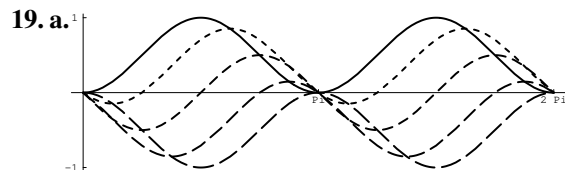
18. b. The period of $f(x)$ is 2π since both $\sin x$ and $\cos x$ have period 2π . The amplitude is approximately 1.22.

18. c. To match $f(x)$ correctly, $A \sin(bx - \varphi)$ must have the same amplitude A and period $2\pi/b$ as f . From part (a) it follows that $A \approx 1.22$ and $b = 1$. (Note: we must determine b , even though the question did not ask that it be done.)

Because $b = 1$, the phase difference φ is the same as the phase shift. Since $y = \sin(x)$ crosses the x -axis at $x = 0$, we will determine the phase shift when we know where $y = f(x)$ crosses the x -axis. But $f(x) = 0$ implies

$$\sin(x) - .7 \cos(x) = 0 \quad \text{or} \quad \tan(x) = .7.$$

We can find $x = \arctan(.7) \approx .61$ (by using a calculator, for example). Thus $\varphi \approx .61$, so $f(x) \approx 1.22 \sin(x - .61)$.



Shown above are the graphs $y = \sin(x) \sin(x - \varphi)$ for $\varphi = 0, \pi/4, \pi/2, 3\pi/4$, and π . The solid curve is $\varphi = 0$; φ increases with the length of the dashes.

19. b. By eye it appears that the average value of y is positive for $\varphi = 0, \pi/4$, negative for $\varphi = 3\pi/4, \pi$ and 0 for $\varphi = \pi/2$.

20. We follow the outline in the exercise. First, use the addition formula

$$\sin(A + B) = \sin(A) \cos(B) + \cos(A) \sin(B)$$

with $A = x$ and $B = -\varphi$:

$$\begin{aligned} \sin(x - \varphi) &= \cos(-\varphi) \sin(x) + \sin(-\varphi) \cos(x) \\ &= \cos(\varphi) \sin(x) - \sin(\varphi) \cos(x). \end{aligned}$$

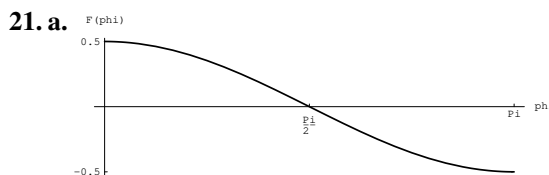
To get the second line, we used the results of exercise 6. Thus, $\sin(x) \sin(x - \varphi) = \cos(\varphi)(\sin(x))^2 - \sin(\varphi) \sin(x) \cos(x)$, so

$$\begin{aligned} F(\varphi) &= \frac{1}{2\pi} \left[\cos(\varphi) \int_0^{2\pi} (\sin(x))^2 dx \right. \\ &\quad \left. - \sin(\varphi) \int_0^{2\pi} \sin(x) \cos(x) dx \right] \end{aligned}$$

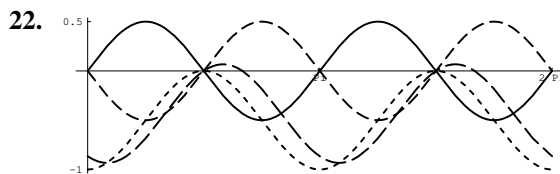
We now calculate the two integrals. From exercise 18 in 6.3 we see $\int_0^{2\pi} (\sin(x))^2 dx = \pi$. Next, since $\sin(2x) = 2 \sin(x) \cos(x)$, we have

$$\int_0^{2\pi} \sin(x) \cos(x) dx = \frac{1}{2} \int_0^{2\pi} \sin(2x) dx = 0$$

because we are integrating over a full period of $\sin(2x)$. Thus, $F(\varphi) = \cos(\varphi)/2$



21. b. $F(0) = 1/2$, $F(\pi/4) = \sqrt{2}/4$, $F(\pi/2) = 0$, $F(3\pi/4) = -\sqrt{2}/4$, $F(\pi) = -1/2$.



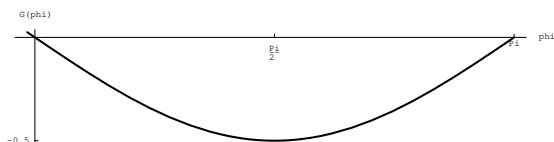
Shown above are the graphs $y = \cos(x) \sin(x - \varphi)$ for $\varphi = 0, \pi/2, 2\pi/3$, and π . The solid curve is $\varphi = 0$; the curve with the shortest dashes is $\varphi = \pi/2$; the curve with the longest dashes is $\varphi = 2\pi/3$; the last (which is the reflection of the solid curve across the x -axis) is $\varphi = \pi$.

By eye it appears that the average value of y is 0 for $\varphi = 0$ and π and negative for $\varphi = \pi/2, 2\pi/3$.

23. a. $\cos(x)\sin(x - \varphi) = \cos(\varphi)\sin(x)\cos(x) - \sin(\varphi)(\cos(x))^2$, implying that

$$G(\varphi) = \frac{1}{2\pi} \left[\cos(\varphi) \int_0^{2\pi} \sin(x)\cos(x) dx - \sin(\varphi) \int_0^{2\pi} (\cos(x))^2 dx \right]$$

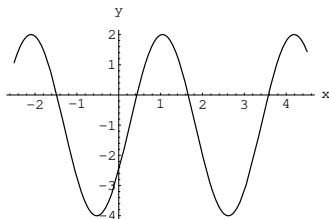
The first integral is 0, as we saw in exercise 20. The second integral is analogous to the integral of $(\sin(x))^2$; they both have the value π . Thus $G(\varphi) = -\sin(\varphi)/2$.



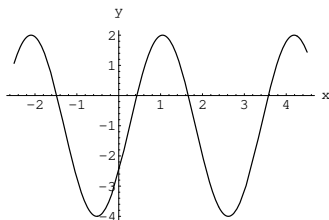
23. b. $G(0) = 0$, $G(\pi/2) = -1/2$, $G(2\pi/3) = -1/4$, $G(\pi) = 0$.

24. The needed phase *difference* is 2π ; this produces a phase *shift* of $2\pi/3$, which causes the graphs to coincide.

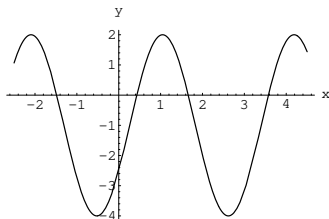
25. a.



25. b.



25. c.



The second and third graphs are identical. In fact, the third comes from the second by a phase shift of π ; but the period of the second (and the third) is π , so a phase shift of π yields a graph identical to what we started with.

26. Since the graph lies between -2 and $+5$, $A = (5 - (-2))/2 = 3.5$. Also, the horizontal “center” of the graph

lies on $y = C = -2 + 3.5 = 1.5$. Since the period is $2 = 2\pi/b$, we see $b = \pi$. Thus, at this stage $G(t) = 3.5 \sin(\pi t - \varphi) + 1.5$.

Finally, we can determine the phase shift $\varphi/b = \varphi/\pi$ by comparing $G(t)$ with the graph of $y = \sin(\pi t)$. Since $y = \sin(\pi t)$ has a minimum at $t = -1/2$ while $y = G(t)$ has its minimum at $t = 0$, we can get G from $\sin(\pi t)$ by carrying out a phase shift of $\varphi/b = 1/2$. Thus $\varphi = \pi/2$.

27. Ad libitum. Here is one possibility, constructed from the fact that $A \sin(b(t - p)) = 0$ when $t = p$, for any A and b :

$$y = 7 + 4 \sin(2\pi(t - 6)/5).$$

Derivatives with Degrees

θ	$\sin(\theta)$	$\sin(\theta)/\theta$
2°	.034899497	.017449748
1°	.017452406	.017452406
$.5^\circ$	.008726536	.017453071

Thus $(\sin(0^\circ))' = .01745 \dots$, so $(\sin(0^\circ))'$ is definitely *not* equal to 1.

28. b. $(\sin(60^\circ))'$ is approximated by $\frac{\sin \theta - \sin(60^\circ)}{\theta - 60^\circ}$

θ	$\frac{\sin \theta - \sqrt{3}/2}{\theta - 60^\circ}$
62°	.0084610945
61°	.0085943034
60.5°	.0086605843
60.2°	.0087002479
60.05°	.0087200500
60.01°	.0087253272

Thus, $(\sin(60^\circ))' = .00872 \dots$

$(\sin(30^\circ))'$ is approximated by $\frac{\sin \theta - \sin(30^\circ)}{\theta - 30^\circ}$

θ	$\frac{\sin \theta - 1/2}{\theta - 30^\circ}$
32°	.014959632
31°	.015038075
30.5°	.015076726
30.2°	.015099733
30.05°	.015111185
30.01°	.015114233

Thus, $(\sin(30^\circ))' = .01511 \dots$; this is definitely not equal to $\cos(30^\circ) = \sqrt{3}/2 \approx .8660254$.

29. Estimate k by evaluating $k = (\sin(\theta))'/\cos(\theta)$ for

$\theta = 0^\circ, 30^\circ$, and 60° .

$$\frac{(\sin 0^\circ)'}{\cos 0^\circ} = (\sin 0^\circ)' \approx .017453$$

$$\frac{(\sin 60^\circ)'}{\cos 60^\circ} = \frac{(\sin 60^\circ)'}{1/2} \approx .017450$$

$$\frac{(\sin 30^\circ)'}{\cos 30^\circ} = \frac{(\sin 30^\circ)'}{\sqrt{3}/2} \approx .017452$$

Thus $k = .01745 \dots$ when θ is measured in degrees.

If θ is the degree measure of an angle and t is its radian measure, then $t = \pi$ when $\theta = 180$. Therefore $\theta = (180/\pi)t$ (and $t = (\pi/180)\theta$).

Now let $\sin(\theta)$ represent the sine of that angle when measured in degrees, and let $f(t)$ represent the sine of the same angle when measured in radians. Then $\sin(\theta) = f(t) = f((\pi/180)\theta)$. Let $\cos(\theta)$ and $g(t)$ be related the same way, so $\cos(\theta) = g(t) = g((\pi/180)\theta)$. We can now differentiate $\sin(\theta)$ using the chain rule:

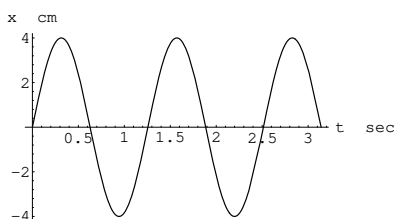
$$(\sin(\theta))' = \left(f\left(\frac{\pi}{180}\theta\right)\right)' = g\left(\frac{\pi}{180}\theta\right) \cdot \frac{\pi}{180} = \frac{\pi}{180} \cos(\theta).$$

This holds for all θ and shows that $k = \pi/180$ exactly.

7.3 Differential Equations with Periodic Solutions

Linear Springs

1. a.

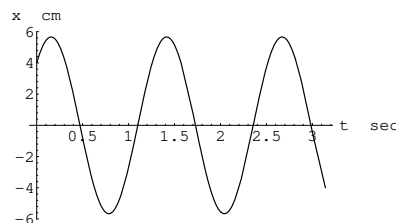


1. b. It appears that $T \approx 1.25$ and $A \approx 4$.

1. c. It appears that the graph is a pure sine function: $x = E \sin(kt)$, for some numbers E and k . Since $x'' = v' = -Ek^2 \sin(bt) = -k^2x$, we see $k = b = 5$. Also, $x' = v = bE \cos(bt)$, so $v(0) = bE = p = 20$. Hence $E = p/b = 4$. Finally, then, $x = 4 \sin(5t)$.

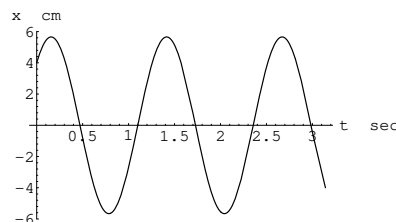
1. d. From the formula, $A = 4$ and $T = 2\pi/b = 2\pi/5 \approx 1.2566$. The theoretical period is thus very close to what the graph shows. According to the calculation, the period T depends *only* on the spring constant b —and not on the impulse p . However, the amplitude satisfies the equation $A = p/b$, so it depends on both the impulse and the spring constant.

2. a.



2. b. It appears that $T \approx 1.25$ and $A = \sqrt{32} \approx 5.6$. The two solutions have the same periods, but the solution in this exercise has the larger amplitude. (It is also phase-shifted to the left by about .157 with respect to the solution in which $p = 0$.)

3.



The periods and amplitudes of the two solutions are the same, but they differ in phase.

4. We have seen already that the spring constant b is the coefficient of t in the formula $x(t) = A \sin(bt - \varphi)$. It remains only to determine A and φ . But

$$a = x(0) = A \sin(-\varphi) = -A \sin(\varphi)$$

$$p = v(0) = bA \cos(-\varphi) = bA \cos(\varphi)$$

Thus $a^2 + (p/b)^2 = A^2$, so $A = \sqrt{a^2 + p^2/b^2} = (\sqrt{a^2 p^2 + p^2})/b$. Also,

$$\frac{ab}{p} = -\frac{\sin(\varphi)}{\cos(\varphi)},$$

so $\varphi = -\arctan(ap/b)$. Thus we have the solution $x(t)$ in terms of all the initial conditions:

$$x(t) = \sin\left(bt + \arctan\left(\frac{ab}{p}\right)\right) \cdot \sqrt{a^2 + \frac{p^2}{b^2}}.$$

5. The text defines $b^2 = c/m$, where c is the spring constant and m is the mass. Suppose c_1 is the constant for the first spring and c_2 is the constant for the second. Let $b_1 = \sqrt{c_1/m}$ and $b_2 = \sqrt{c_2/m}$. We know the period T of a spring is related to b by $T = 2\pi/b$ and the frequency is $1/T = b/2\pi$. Therefore the frequencies of the two springs are

$$\text{freq}_1 = \frac{\sqrt{c_1}}{2\pi\sqrt{m}} \quad \text{freq}_2 = \frac{\sqrt{c_2}}{2\pi\sqrt{m}}.$$

We are told $c_2 = 2c_1$, so $\sqrt{c_2} = \sqrt{2}\sqrt{c_1}$ and hence

$$\text{freq}_2 = \sqrt{2} \frac{\sqrt{c_1}}{2\pi\sqrt{m}} = \sqrt{2} \cdot \text{freq}_1.$$

Thus, *the second spring oscillates $\sqrt{2}$ times as fast as the first.* In other words, if two springs carry the same mass, the “tighter” one oscillates faster.

6. a. By analogy with the solution to the previous exercise, we write

$$\text{freq}_1 = \frac{\sqrt{c}}{2\pi} \frac{1}{\sqrt{m_1}} \quad \text{freq}_2 = \frac{\sqrt{c}}{2\pi} \frac{1}{\sqrt{m_2}},$$

where m_1 and m_2 are the masses on the first and second springs. We are told $m_2 = 2m_1$, so $\sqrt{m_2} = \sqrt{2}\sqrt{m_1}$ and hence

$$\text{freq}_2 = \frac{\sqrt{c}}{2\pi} \frac{1}{\sqrt{2}\sqrt{m_1}} = \frac{1}{\sqrt{2}} \cdot \text{freq}_1.$$

Thus, $\text{freq}_1 = \sqrt{2} \cdot \text{freq}_2$, so *the first spring oscillates with the higher frequency, and it is higher by the factor $\sqrt{2}$.*

6. b. According to part (a), the frequency will be higher when the mass is lighter. If the mass is cut in half, the frequency increases by the factor $\sqrt{2}$. Therefore, to increase the frequency by the factor 2, the mass must be cut in half *twice*. Since the original mass was 200 grams, a mass of 50 grams will oscillate at twice the frequency.

The reality check. It is reasonably intuitive that a tighter spring oscillates faster while a heavier weight oscillates more slowly.

7. a. To show $E = \frac{1}{2}(v(t))^2 + \frac{1}{2}b^2(x(t))^2$ doesn't vary with t , we compute the derivative of E with respect to t . This requires the chain rule:

$$\begin{aligned} E' &= v(t)v'(t) + b^2x(t)x'(t) \\ &= v(t) \cdot -b^2x(t) + b^2x(t)v(t) = 0. \end{aligned}$$

At the last step we used the differential equations $x' = v$, $v' = -b^2x$.

7. b. Since E has the same value for any t , we can find E by setting $t = 0$ in the formula for E (and using $x(0) = a$, $v(0) = p$): $E = \frac{1}{2}p^2 + \frac{1}{2}b^2a^2$.

7. c. The units for E are cm^2 per sec^2 .

8. a. If $x = 0$, then $E = \frac{1}{2}v^2 = \frac{1}{2}p^2 + \frac{1}{2}b^2a^2$. Therefore, $v = \pm\sqrt{p^2 + b^2a^2}$.

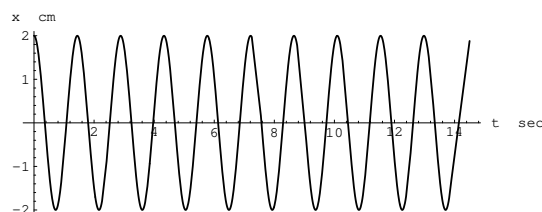
8. b. At a moment when the weight is motionless, $v = 0$. But if $v = 0$ then $E = \frac{1}{2}b^2x^2 = \frac{1}{2}p^2 + \frac{1}{2}b^2a^2$. Therefore

$x = \pm\sqrt{a^2 + p^2/b^2}$. This tells us how far the weight is from the rest position.

9. Whenever $v = 0$, we have $x = \pm\sqrt{p^2/b^2 + a^2}$. If we take $v = 0$ and $x = +\sqrt{p^2/b^2 + a^2}$ as new initial conditions, we see that the spring will, at some later time t_1 , return to these values, so the motion will repeat itself exactly.

Nonlinear Springs

10. a. There are no formulas available; we must make estimates based on numerical methods. We use a numerical differential equation solver to plot the solution to the initial value problem over ten periods.

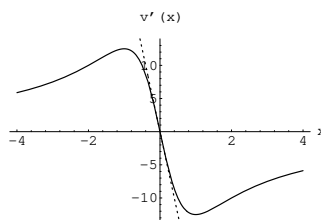


In ten periods, $\Delta t \approx 14.4$; therefore we estimate the period is $T = 1.44$ sec. and the frequency is $1/1.44 = .69$ cycles per sec. [Note: plotting ten periods is a good practical measure to increase accuracy.]

10. b. To double the frequency we must cut the period in half: $T \approx .72$. By trial and error using a numerical differential equation solver, we find the initial displacement must be $a \approx 9.1$. [Note: for reference $a = 9$ produces $T \approx .726$; while $a = 9.2$ produces $T \approx .714$.]

11. Since the amplitude is very small, the non-linear term is negligible and the spring will be described by the linear part of the differential equation: $x' = v$, $v' = -16x$. Thus $b = 4$ and the frequency is $b/2\pi = 4/2\pi = 2/\pi$ cycles per second.

12. a. As the graph below shows, the acceleration function is *less* than the linear function that agrees with it at the origin. Under these conditions, the spring is a soft spring, by definition.



The slope of the acceleration function at the origin is -25 ; therefore, the linear function that is sketched is $f(x) = -25x$.

12. b. Using a numerical differential equation solver, we find that the frequency is 10 cycles/41.90 sec = .24 cycles/sec.

12. c. Now the frequency is $10/83.87 = .12$ cycles/sec. When the displacement is doubled the frequency is roughly cut in half.

12. d. Now the frequency is $4/339.5 = .01$ cycles/sec. When the displacement is increased by a factor of 25 the frequency is reduced to 1/24-th of its original size.

13. As in exercise 7, we differentiate E with respect to t using the chain rule:

$$\begin{aligned} E' &= v \cdot v' + b^2 x \cdot x' + \beta x^3 \cdot x' \\ &= v \cdot (-b^2 x - \beta x^3) + b^2 x \cdot v + \beta x^3 \cdot v \\ &= 0. \end{aligned}$$

14. a. $E = \frac{1}{2}0^2 + 8 \cdot 2^2 + \frac{1}{4} \cdot 2^4 = 36$.

14. b. At the moment the spring passes through the rest position, $x = 0$ and E reduces to $E = \frac{1}{2}v^2 = 36$. Therefore $v = \pm\sqrt{72}$.

14. c. The size of x is limited by the fact that E is fixed and $E = \frac{1}{2}v^2 + 8x^2 + \frac{1}{4}x^4$. To make x as large as possible we should set $v = 0$; then

$$36 = E = 8x^2 + \frac{1}{4}x^4 \text{ or } x^4 + 32x^2 - 144 = 0.$$

The last equation factors as $(x^2 + 36)(x^2 - 4) = 0$, implying that $x^2 = 4$ or $x = \pm 2$ cm. These are the largest values that x can attain.

15. Define $E(x, v) = \frac{1}{2}v^2 + \frac{1}{2}b^2x^2 + \frac{1}{4}\beta x^4$. In exercise 13 we showed that this is a first integral for the given system of differential equations. Because the initial condition are $x(0) = a$, $v(0) = p$, the first integral has the value $E = \frac{1}{2}p^2 + \frac{1}{2}b^2a^2 + \frac{1}{4}\beta a^4$.

The argument that the solution $(x(t), v(t))$ is periodic is essentially the same as the argument in exercise 7; in both cases the key is the presence of a first integral.

16. a. We know from Exercise 12 that $x' = v$ and $v' = -25x/(1+x^2)$. Therefore,

$$\begin{aligned} \frac{d}{dt}E(x(t), v(t)) &= v \cdot v' + \frac{25}{2} \frac{2x}{1+x^2} \cdot x' \\ &= v \cdot \frac{-25x}{1+x^2} + \frac{25x}{1+x^2} \cdot v \\ &= 0. \end{aligned}$$

(Notice that we used the chain rule here: $(\frac{1}{2}(v(t))^2)' = v(t)v'(t)$.) Thus E is a first integral for the system.

16. b. The initial conditions fix $E = \frac{25}{2} \ln(17)$. When the weight moves past the rest position, $x = 0$. Therefore,

$$\frac{25}{2} \ln(17) = E = \frac{1}{2}v^2 + 0,$$

implying $v = \pm 5\sqrt{\ln(17)}$. So the speed of the weight is $5\sqrt{\ln 17}$, or about 8.416 cm/sec.

16. c. Because the motion has a first integral, it is periodic, and the argument is the same as in exercises 7 and 15.

17. In the other cases where a first integral existed, v' was a function of x alone and E contained the negative of the antiderivative of that function. That suggests we set

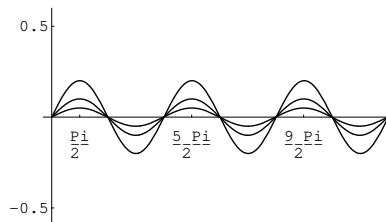
$$E(x, v) = v^2 + \int^x f(x) dx.$$

(We could multiply everything by $\frac{1}{2}$, but either way we'll get a first integral.) To see that E is a first integral, we differentiate $E(x(t), v(t))$ with respect to t when $(x(t), v(t))$ is a solution to the system $x' = v$, $v' = -f(x)$:

$$\begin{aligned} \frac{d}{dt}E(x(t), v(t)) &= 2v \cdot v' + f(x) \cdot x' \\ &= v \cdot -f(x) + f(x) \cdot v \\ &= 0. \end{aligned}$$

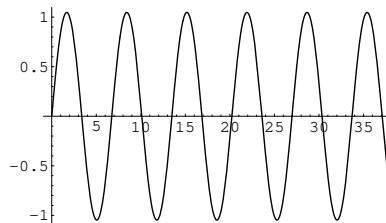
The Pendulum

18.

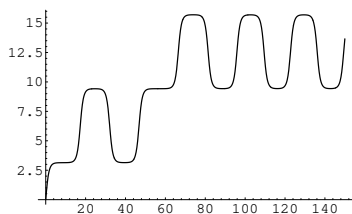


The largest initial impulse p yields the solution with the largest amplitude. All three solutions have about the same period: 6.28 seconds.

19. According to the graph below, the solution with initial impulse $p = 1$ completes 5 cycles in about 33.8 seconds, so its period is about 6.75 seconds.



Numerical solutions when $p = 2$ are almost always wrong (for reasons explored in the immediately following exercises). The true solution approaches the horizontal line $x = \pi$ asymptotically. The following is one (wrong) answer generated by a differential equation solver (Mathematica).



20. Trial and error reveals that the threshold value for p is 2.

21. a. By trial and error we find the threshold value for p is about 1.4.

21. b. Once again, by trial and error we find the threshold $p \approx -1.4$. Thus, the smallest $|p|$ that causes spinning is about 1.4 when the pendulum starts in a horizontal position.

22. The first integral is $E(x, v) = 1 - \cos(x) + \frac{1}{2}v^2$. The initial conditions force $E = 1 - \cos(a)$. Therefore, when $x = 0$, $1 - \cos(a) = \frac{1}{2}v^2$, so $v = \pm\sqrt{2 - 2\cos(a)}$.

23. a. Now $E = \frac{1}{2}p^2$.

23. b. If the pendulum comes to rest exactly at the top of its swing, then $x = \pi$ and $v = 0$. We know E has the same value everywhere, so

$$\frac{1}{2}p^2 = E(\pi, 0) = 1 - \cos(\pi) + \frac{1}{2}v^2 = 2.$$

Therefore $p^2 = 4$ so $p = \pm 2$. This is the initial impulse that will push the pendulum just to the top when $x(0) = 0$. Therefore any impulse $|p| > 2$ will push it “over the top”.

24. The initial conditions force $E = E(\pi/2, 0) = 1 + \frac{1}{2}p^2$. If the pendulum is motionless at the top (as in the previous exercise), then

$$1 + \frac{1}{2}p^2 = E(\pi, 0) = 1 - \cos(\pi) + \frac{1}{2}v^2 = 2.$$

Therefore $p^2 = 2$ so $p = \pm\sqrt{2}$. Thus any impulse $|p| > \sqrt{2}$ will push the pendulum over the top.

Predator-Prey Ecology

25. a. When $x(t) = 10$, $y(t) = 0$, then $x'(t) = y'(t) = 0$. We must now substitute these values into the differential

equations. The result is

$$0 = .6 \cdot 10(1 - 10/10) - .5 \cdot 10 \cdot 0/11 = 0$$

$$0 = .1 \cdot 0(1 - 0/20) = 0.$$

The right-hand sides are indeed 0, so the given functions are solutions.

25. b. When $x(t) = 0$, y' is undefined, so $x(t) = 0$, $y(t) = 0$ is not an equilibrium solution.

25. c. Here is a derivation; this demonstrates that the given expressions are a solution. We want to find x and y which make the right-hand sides of the differential equations equal 0:

$$0 = .6x(1 - x/10) - .5xy/(x + 1)$$

$$0 = .1y(1 - y/2x).$$

Since $x = 0$ is ruled out and since $y = 0$ forces $x = 10$, we are looking for *non-zero* solutions x and y ; thus we can divide the first equation by $10x$ and the second by $.1y$:

$$0 = 6(1 - x/10) - 5y/(x + 1)$$

$$0 = 1 - y/2x.$$

The second equation implies $y = 2x$. Substituting this value for y into the first equation, we get

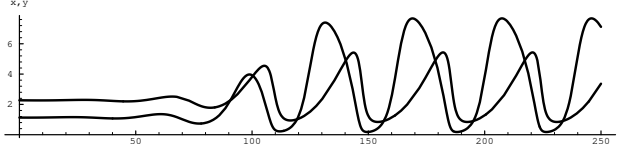
$$0 = 6 - .6x - 10x/(x + 1),$$

This reduces to $3x^2 + 23x - 30 = 0$, so

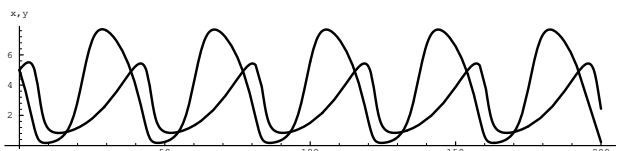
$$x = \frac{-23 \pm \sqrt{23^2 + 4 \cdot 3 \cdot 30}}{6} = \frac{-23 \pm \sqrt{889}}{6},$$

and y is twice this value.

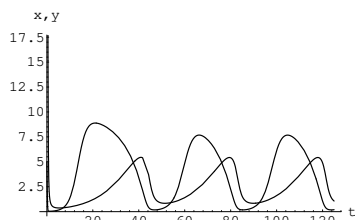
26. a. The solutions (see below) drift away from the equilibrium values, so we infer the equilibrium is *unstable*.



26. b. The graphs below have the same shapes, the same periods, and the same spacing between peaks as those in the text.

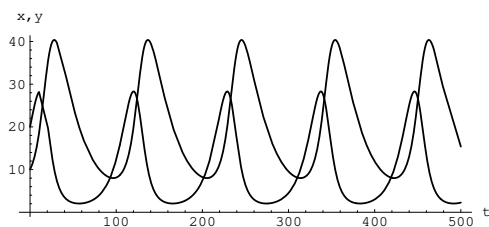


27.

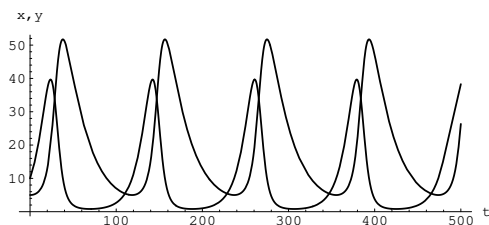


The pest *does not* disappear. Instead, y plunges rapidly to a small value, but then recovers and the eventual solution looks like all the others. There was no ultimate benefit in starting with a very large value for y .

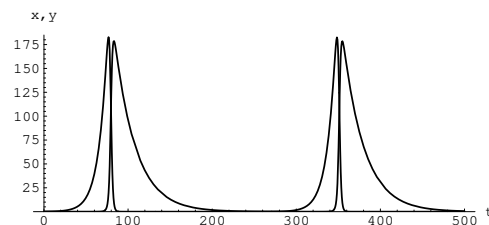
28. The prey population (see next page) has its first peak at $t = 11.6$ and its second at $t = 120.4$, so its period is 108.8 units. The predator has its first peak at $t = 27.6$ and its second at $t = 136.4$. Its period is therefore also 108.8 units. The prey peaks 16 units before the predator.



29. The prey population has its first peak at $t = 24$ and its second at $t = 142.4$, so its period is 118.4 units. The predator has its first peak at $t = 38.4$ and its second at $t = 156.8$. Its period is therefore also 118.4 units. The prey peaks 14.4 units before the predator. The periods and peak spacing are different from those in the previous exercise; in the Lotka-Volterra model, initial conditions *do* affect the period (and amplitude) of solutions.



30. If $x(t) = 0$ and $y(t) = 0$, then $x' = 0$ and $y' = 0$. If we substitute these values into the Lotka-Volterra equation, both sides are 0, so $x = y = 0$ is a solution. However, this equilibrium is *unstable*, because the solution graphed below has nearby initial conditions but fails to stay close to $x = y = 0$ over time. (The periods and amplitudes are quite large, and the solutions return periodically to values near 0.)

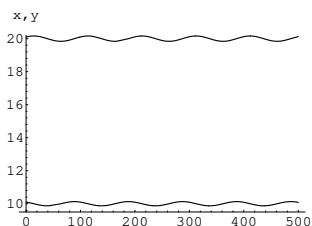


31. If $x(t) = 10$ and $y(t) = 20$ then $x' = 0$ and $y' = 0$. Substituting these values into the Lotka-Volterra equations, we get

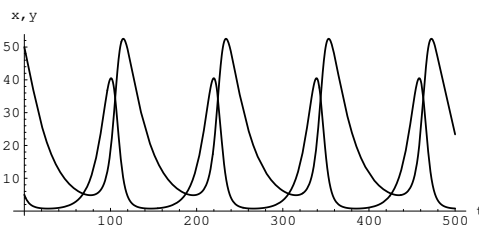
$$0 = .1 \cdot 10 - .005 \cdot 10 \cdot 20 = 1 - 1 = 0$$

$$0 = .004 \cdot 10 \cdot 20 - .04 \cdot 20 = .8 - .8 = 0$$

To test whether this is a stable equilibrium, we plot the solution with the nearby initial conditions $x(0) = 10.1$ and $y(0) = 20.1$. The solutions stay “close to” the equilibrium solutions—a reasonable indication of stability (known as *neutral stability*). v



32. The pest does not disappear. On the contrary, a periodic pattern develops immediately and continues thereafter.



33. As in previous exercises, we must show that the derivative with respect to t of $E(x(t), y(t))$ is 0 when $x(t)$ and $y(t)$ are solutions of the differential equations.

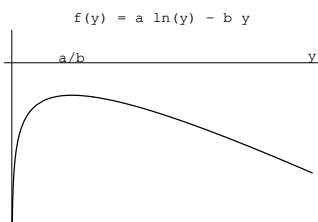
$$\begin{aligned} \frac{d}{dt} E(x(t), y(t)) &= \frac{a}{y} y' + \frac{d}{x} x' - b y' - c x' = \\ &= \frac{a}{y} (cxy - dy) + \frac{d}{x} (ax - bxy) - b(cxy - dy) \\ &\quad - c(ax - bxy) \\ &= acx - ad + da - dby - bcxy + bdy - cax + cbxy \\ &= 0. \end{aligned}$$

34. The proof of periodicity follows the pattern of other examples that have a first integral. We assume there is already evidence (from numerical differential equation solvers, for instance) that the solution *may* be periodic. The task, then, is to show that the evidence is not illusory. Here is what this means. Take the solution that satisfies the initial conditions $x(0) = p$, $y(0) = q$. Suppose that, at some later time t_1 , x has returned to p (i.e., $x(t_1) = p$) and y has returned to a value that is *close* to q . We must show that $y(t_1) = q$ exactly. Uniqueness of solutions to initial value problems will then guarantee that $x(t)$ and $y(t)$ repeat themselves forever.

We know $E = a \ln(q) + d \ln(p) - bq - cp$ for the given solution; E has the same value when $t = t_1$:

$$\begin{aligned} a \ln(q) + d \ln(p) - bq - cp &= \\ a \ln(y(t_1)) + d \ln(p) - by(t_1) - cp & \\ a \ln(q) - bq = a \ln(y(t_1)) - by(t_1). \end{aligned} \quad (7.1)$$

It comes down to this: does the last equation imply $y(t_1) = q$, assuming that $y(t_1)$ is already close to q ? In fact, it does. Consider the graph of the function $f(y) = a \ln(y) - by$:

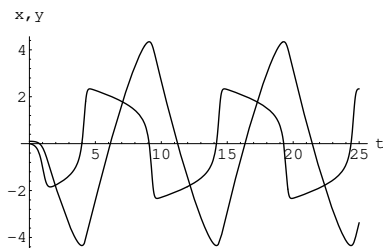


Since $f'(y) = a/y - b$, $f' = 0$ when $y = a/b$. This is the maximum point on the graph. If $q = a/b$ then $y = a/b$ as well, because there is no other y value for which $f(y) = f(q)$. If $q \neq a/b$, then $f'(q) \neq 0$. This implies f is one-to-one near q , so once again there is only one y for which $f(y) = f(q)$ when y is near q .

By what has already been said, this shows the solution we started with is periodic.

The Van der Pol Oscillator

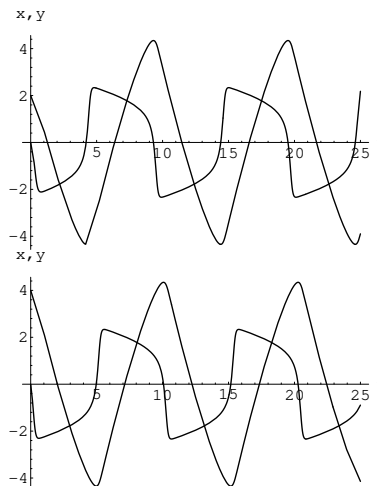
35.



The squatter, “squarer” graph is $y(t)$. Its first lower lobe is different from the later ones; this shows that $y(t)$ is not initially periodic.

The (eventual) amplitude of x is about 4.3; the amplitude of y is about 2.3; their periods are both about 10.2. (One way to get good estimates for the periods is to measure the interval covered by 10 cycles and then divide by 10.)

36.



$x(0) = 2$ in the upper graph and $x(0) = 4$ in the lower. In all cases the period is 10.2, the amplitude of x is 4.3, and the amplitude of y is 2.3. The initial current $x(0)$ has no effect on the eventual outcome (though it does appear to affect the amount of time it takes the solutions to settle into periodic behavior). The period and amplitudes of the solutions are independent of the initial conditions.

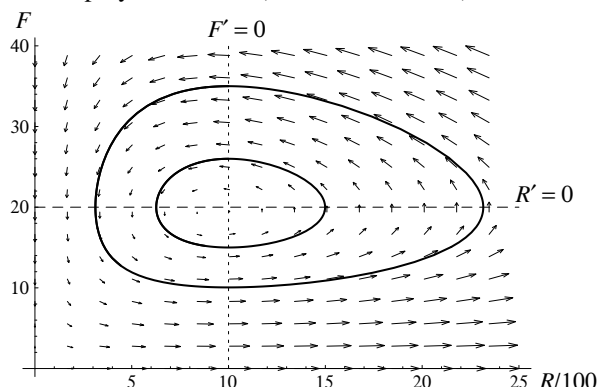
Solutions: Chapter 8

Dynamical Systems

8.1 State Spaces and Vector Fields

Technical note: We produced most of the graphics with Mathematica 3. Since vectors in a vector field can vary in length by several orders of magnitude, we rescaled them to improve visibility. Of course this distorts their relative size. To rescale, we typically used the option `ScaleFunction->(#/(#+a)&)` (with `a` equal to 1, .1, or .01) in `PlotVectorField`. Additionally, we used `ScaleFactor` to adjust lengths overall. Finally, the first exercise presents a further problem, because the ranges of R and F differ by two orders of magnitude. Mathematica has trouble plotting the vector field in these circumstances, so we rescaled R as $r = R/100$ and modified the differential equations accordingly.

1. a. • This model is a simplification of the one in the text: the carrying capacity for rabbits has been removed. All that remains are the inherent growth rates of the individual species (in the terms aR and $-dF$) and the predator-prey interaction (in $-bRF$ and cRF).



• $R' = 0$ on the horizontal line $F = a/b = 20$ (shown dashed in the figure above). $R' > 0$ below the line and $R' < 0$ above it. $F' = 0$ on the vertical line $R = d/c = 1000$ (shown dotted); $F' < 0$ to the left and $F' > 0$ to the right.

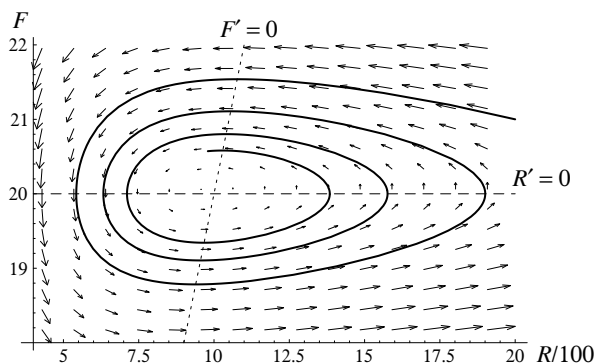
(R' is also 0 on the F -axis and F' is 0 on the R -axis.)

• In the first quadrant there is only one equilibrium: $(R, F) = (1000, 20)$. The origin is also an equilibrium. The equilibrium points are precisely those that are both red (here, dashed) and green (dotted).

• Other trajectories are ovals that enclose the equilibrium point.

• Because trajectories neither approach nor recede from the equilibrium $(20, 1000)$, we can say it is *neutral*. The origin, however, is *unstable*, because trajectories flow away from it.

1. b. • In this model, R' is the same as in the standard Lotka-Volterra model. F' is essentially the same as in the May model from chapter 4 (pp. 168–170) and chapter 7 (pp. 394–396): R plays the role of a carrying capacity for the fox population.

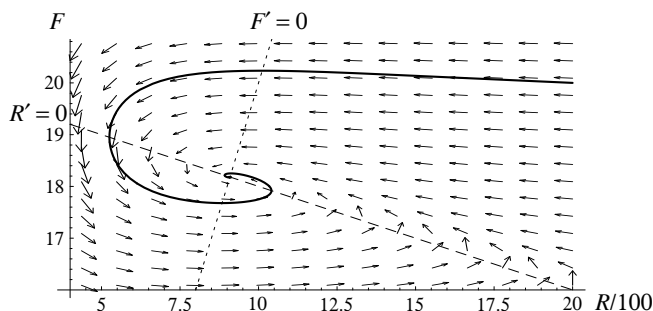


• $R' = 0$ on the horizontal line $F = a/b = 20$ (shown dashed in the figure above). Note the change in the vertical scale! $R' > 0$ below the line and $R' < 0$ above it. $F' = 0$ on the sloping line $F = \frac{c}{f}R = .02R$ (shown dotted); $F' < 0$ to the left of this line and $F' > 0$ to the right. (R' is also 0 on the F -axis—but F' is undefined there!—and F' is 0 on the R -axis.)

• In the first quadrant there is only one equilibrium: $(R, F) = (1000, 20)$. The equilibrium points are precisely those that are both red and green.

- Trajectories spiral in to the equilibrium point.
- Because trajectories spiral in to the equilibrium (20, 1000), we can say it is *stable*.

1. c. • This model combines the R' equation from the Lotka-Volterra model that has a carrying capacity for rabbits with the Leslie-Gower model's F' equation.

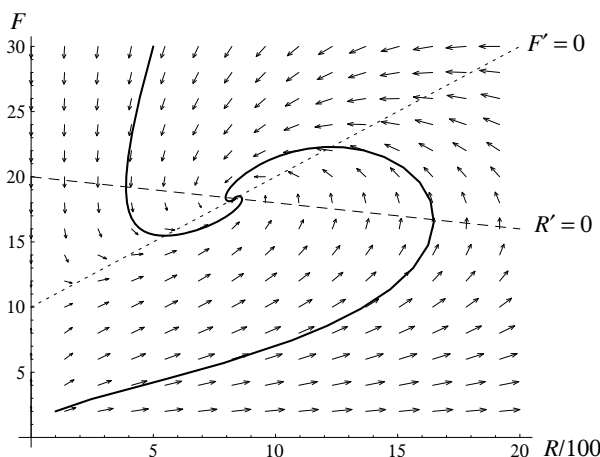


- $R' = 0$ on the sloping line $F = a/b - aR/bK = 20 - .002R$ (shown dashed). $R' > 0$ below the line and $R' < 0$ above it. $F' = 0$ on the sloping line $F = \frac{g}{h}R = .02R$ (dotted); $F' < 0$ to the left of this line and $F' > 0$ to the right. (R' is also 0 on the F -axis—but F' is undefined there!—and F' is 0 on the R -axis.)

- In the first quadrant there is only one equilibrium: $(R, F) = (10000/9, 200/9) \approx (1111.1, 22.2)$. The equilibrium points are precisely those that are both red and green.

- Trajectories spiral in to the equilibrium point.
- Because trajectories spiral in to the equilibrium (200/9, 10000/9), we can say it is *stable*.

1. d. • This model uses the R' equation from the last example. We can consider the F' equation to be a modification of the F' in the original Lotka-Volterra model. The change is in what is assumed to happen to foxes in the absence of rabbits.



In the original Lotka-Volterra model, foxes just died out (via $-dF$). In the present equation, we can write the corresponding term as $gF - hF^2 = gF(1 - F/L)$, where $L = g/h$. This shows that foxes can now grow, but their growth is limited by a carrying capacity factor L .

- $R' = 0$ on the sloping line $F = a/b - aR/bK = 20 - .002R$ (dashed). $R' > 0$ below the line and $R' < 0$ above it. $F' = 0$ on the sloping line $F = \frac{g}{h} + \frac{c}{h}R = 10 + .01R$ (dotted); $F' < 0$ above this line and $F' > 0$ below it. (R' is also 0 on the F -axis and F' is 0 on the R -axis.)

- Inside the first quadrant there is only one equilibrium: $(R, F) = (2500/3, 55/3) \approx (833.3, 18.3)$; $(0, 10)$ is also. The equilibrium points are precisely those that are both red and green.

- Trajectories spiral in to the equilibrium point.
- Because trajectories spiral in to the equilibrium (2500/3, 55/3), we can say it is *stable*.

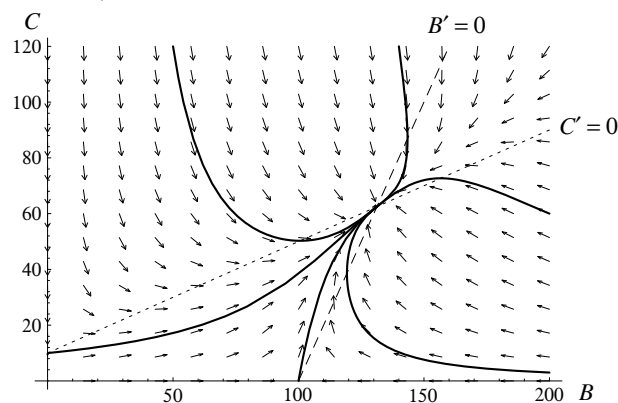
2. a. The key feature of symbiosis is that the interaction between species benefits *both* species. In this model, interaction is conveyed by the BC term (in each differential equation). For there to be symbiosis, this term must have a positive coefficient in both equations. It does, so the equations do describe symbiosis.

2. b. If we write the equations in the form

$$B' = .1B(1 - B/100) + .005BC$$

$$C' = .03C(1 - C/10) + .04BC,$$

we can see that the negative term in each equation plays the role of a carrying capacity term; that is, it represents a natural limit to the size of each population (in the absence of the other).

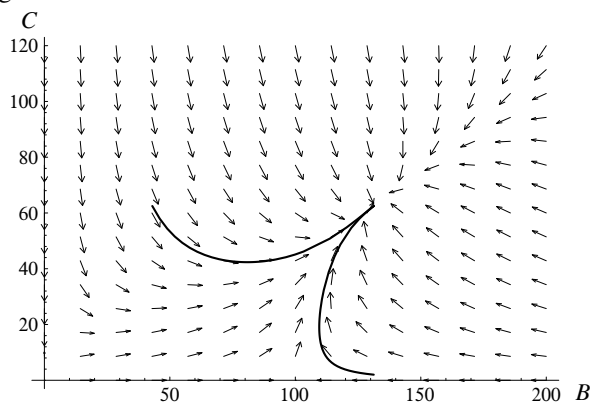


2. c. The equilibrium points are the points where $B' = 0$ and $C' = 0$ intersect. $B' = 0$ on the C -axis and on the line $C = 2B - 200$ (shown dashed in the figure above); $C' = 0$ on the B -axis and on the line $C = .4B + 10$ (dotted). $B' > 0$ to the left of the dashed line, and $B' < 0$ to the right; $C' > 0$ below the dotted line, and $C' < 0$ above. Equilibrium points: $(B, C) = (0, 0)$, $(100, 0)$, $(0, 10)$, $(131.25, 62.5)$.

2. d. All the trajectories shown move toward the equilibrium at $(131.25, 62.5)$. This includes one that moves away from the equilibrium on the B -axis at $(100, 0)$ and another that moves away from the equilibrium on the C -axis at $(0, 10)$. Only $(131.25, 62.5)$ is stable. The two equilibria on the axes are saddles: trajectories that begin on that axis approach that point, but trajectories that begin off the axis flow away. The origin is a repeller.

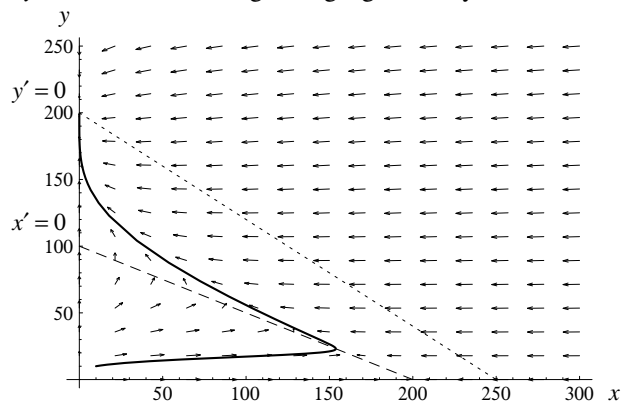
2. e. We have $B = 20$ and $C = 10$. The lower left trajectory in the previous figure shows what happens: the point (B, C) moves to the equilibrium $(131.25, 62.5)$. In words, the bee population grows to 13,125 and the clover to 62,500 pounds per acre.

2. f. After the field has been mown, $B = 131.25$ but $C = 2$. The bee population drops temporarily as the clover recovers with both populations returning to the equilibrium values $B = 131.25$, $C = 62.5$. One of the trajectories in the figure below shows this.



2. g. After the insecticide application, $C = 62.5$ but $B = 43$. The figure above shows that, in response, the clover population initially declines while the bee population recovers. (It begins to increase after the bees reach a population of about 8000.) Eventually, both populations return to the equilibrium values $B = 131.25$, $C = 62.5$.

3. a. When $x = y = 10$, $x' = 1.275$ and $y' = .273$, so x' is about 5 times as large as y' . At $t = .5$, $x \approx 10.6375$ while $y \approx 10.1365$, so x is growing significantly faster than y .

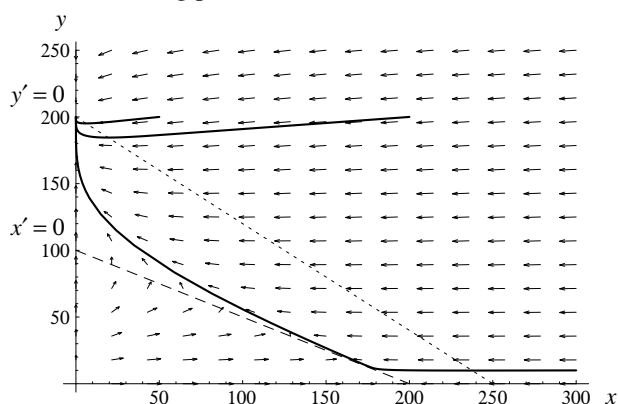


3. b. There are three equilibrium points at $(0, 0)$, $(200, 0)$, and $(0, 200)$.

3. c. In the previous figure, $x' = 0$ on the dashed line and $y' = 0$ on the dotted line. The lines don't intersect, but they divide the first quadrant into three regions. Nearest the origin, vectors point up and to the right; next, up and to the left; last (outermost), down and to the left.

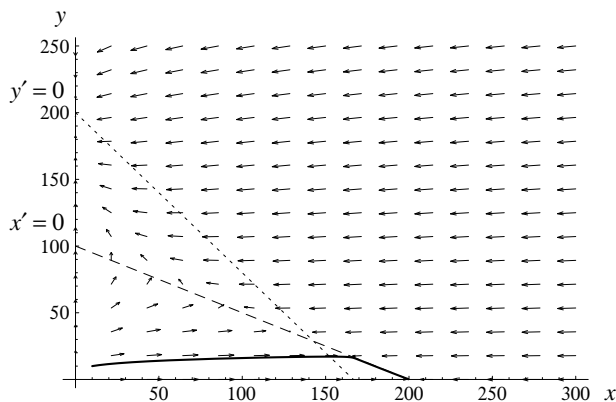
3. d. The trajectory is shown in the previous figure. It shows that x grows rapidly to about 150 while y grows only to about 25. After that x declines but y continues to grow. Eventually y approaches 200 and x approaches 0.

3. e. The figure below shows that all trajectories go to the equilibrium at $(0, 200)$. Notice that the trajectory that starts at $(300, 10)$ passes through—or near to—the point $(150, 25)$; consequently, it follows the same path as the trajectory that starts at that point. On the basis of this evidence, all starting points lead to the same outcome.



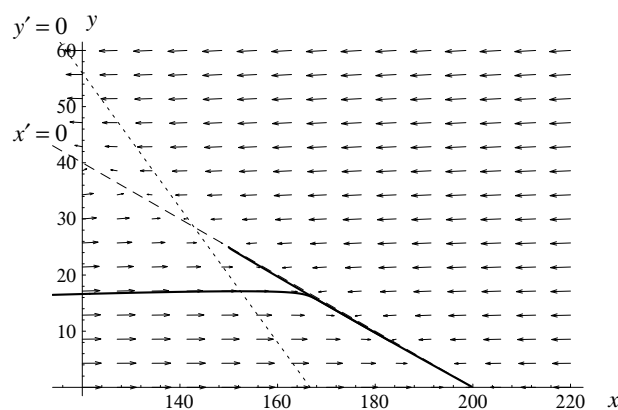
3. f. $(0, 200)$ is the only attractor; $(200, 0)$ is a saddle point, and $(0, 0)$ is a repeller.

4. a. Compare this figure with the figure in exercise 3c. The dotted line $y' = 0$ still passes through $(0, 200)$, but it is steeper now, and intersects the dashed line $x' = 0$ in the first quadrant. This intersection is a new equilibrium point; it is at $x = (1000/7, 1000/35) \approx (142, 28)$. In the fourth zone created by this change, the vector field points down and to the right. (The three equilibrium points from exercise 3 continue to appear.)



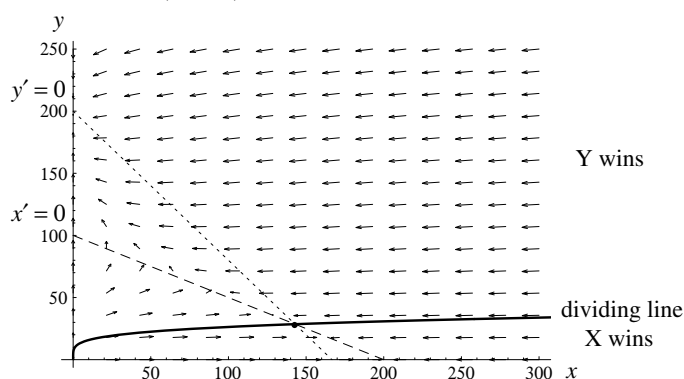
4. b. The outcome for initial conditions $x = y = 10$ is significantly different from what it was in exercise 3: this time x goes to 200 and y disappears.

4. c. The figure below, which is a magnification of the previous one, shows that the initial conditions $x = 150$, $y = 25$ also lead to the equilibrium $(200, 0)$. This outcome, too, is significantly different from what it was in exercise 3. These two results show that X can now win—at least with some initial conditions.



4. d. X and Y can coexist at the equilibrium $(x, y) = (1000/7, 1000/35)$. However, this equilibrium is unstable. If x and y are disturbed even slightly from $(1000/7, 1000/35)$, they gradually drift away from this equilibrium. Eventually, either x or y becomes 0 and one species disappears entirely.

4. e. The previous parts suggest there is a “dividing line” (a curve, actually) consisting of the trajectories that go in to the new equilibrium at $(1000/7, 1000/35)$. Initial conditions below this curve are attracted to $(200, 0)$; those above, to $(0, 200)$.



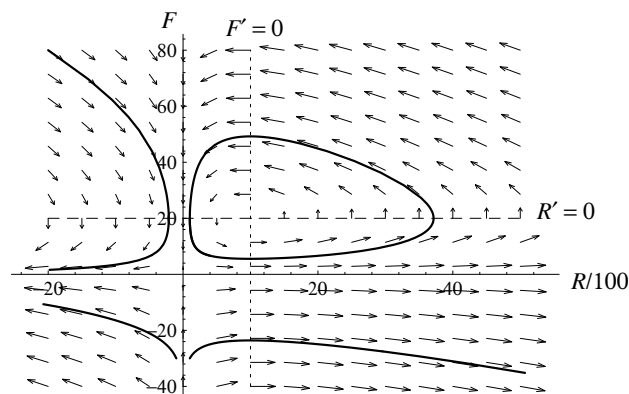
Drawing these trajectories presents a technical problem: it is virtually impossible to choose the initial point so the trajectory doesn't simply go to one of the two attractors. One way out of this is to “turn around” the vector field (so the attractors become repellers). Then we can

get the dividing curve by choosing initial conditions just to the left of the equilibrium point and just to the right. That is how the figure above is constructed; the dividing curve looks parabolic, and goes through the origin and the new equilibrium.

4. f. The equilibrium points $(0, 200)$ and $(200, 0)$ are stable, while the equilibrium points $(0, 0)$ and $(1000/7, 1000/35)$ are unstable.

4. g. This model is consistent with the *principle of competitive exclusion*. In the long run, the system finds its way to a stable equilibrium. There are two, but X is missing at one of them and Y is missing at the other: X and Y do not coexist in *stable* equilibrium. (Of course, they do coexist at the equilibrium inside the first quadrant—but that is not stable, and attempts to get numerical solutions to reach that equilibrium help persuade us that that equilibrium will not be achieved in nature!)

5. a. Trajectories that start with $R < 0$ approach $(R, F) = (-\infty, 0)$; those that start with $R > 0$ and $F < 0$ approach $(\infty, -\infty)$. The trajectories in the following figure have starting points of $(100, 20)$, $(-2000, 80)$, $(-100, -30)$ and $(100, -30)$.



5. b. The Euler “trajectory” spirals outward; see the figure on the next page. The problem arises because vectors in the vector field are *tangent* to the true, closed-loop, trajectories, and the Euler approximation carries us along a vector for some finite distance. Since the loops are convex everywhere, the very first step of the Euler approximation takes us outside the closed loop. Each successive step carries us further and further; we never get back to the closed loop. When Δt is reduced to about .01, the Euler trajectory appears (on a VGA graphics screen) to close after a *single* loop. (Of course, if you let the trajectory continue, it will eventually diverge noticeably from the true closed-loop trajectory.)

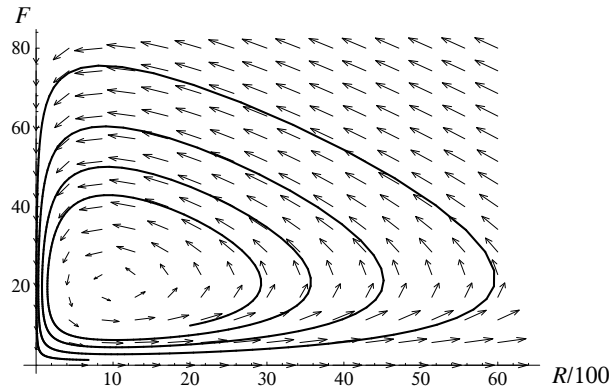
Here is a QuickBasic program that plots the trajectory over about five complete cycles.

```

DEFDBL A-Z
SCREEN 9
COLOR 8, 7
WINDOW (0, 0)-(4000, 50)
R = 2000
F = 10
deltat = .01
FOR k = 0 TO 55000
  Rprime = .1 * R - .005 * R * F
  Fprime = .00004 * R * F - .04 * F
  deltaR = Rprime * deltat
  deltaF = Fprime * deltat
  j = 2 * (k \ 10880) + 6
  LINE (R, F)-(R + deltaR, F + deltaF), j
  R = R + deltaR
  F = F + deltaF
NEXT k

```

The line defining j uses “integer division”: $A \setminus B$ is the integer part of the quotient A/B . The number 10880 is chosen because it takes about 10880 steps for the trajectory to complete one loop. Thus, the trajectory changes color for each loop, making it easier to see how much the next cycle differs from the previous.



5. c. The program stops—the computation “blows up”. By modifying the program above to print values of R and F , we can see what goes wrong. First of all, the large step size allows the trajectory to “escape from” the first quadrant. This first happens when $k = 16$; then $R = -95.688$ and $F = 131.752$. On my machine, the last step happens when $k = 214$. The trajectory has found its way into the fourth quadrant. Here are the last few steps (giving only the order of magnitude of R and F):

k	R	F
211	10^{20}	-10^{18}
212	10^{37}	-10^{35}
213	10^{70}	-10^{68}
214	10^{136}	-10^{134}

The numbers change so much because $R' = aR - bRF$ is essentially the same order of magnitude as bRF . At the

next step ($k = 215$), $R \sim 10^{-3} \times 10^{136} \times 10^{134} \sim 10^{267}$, and this is apparently too big for my machine!

5. d. Divide $R' = R(a - bF)$ by $R(a - bF)$; the result is 1. The same is true if we divide $F' = F(cR - d)$ by $F(cR - d)$. Therefore,

$$\frac{R'}{R(a - bF)} = \frac{F'}{F(cR - d)}, \text{ or } \frac{R'}{R} \frac{1}{(a - bF)} = \frac{F'}{F} \frac{1}{(cR - d)},$$

which is the first thing to show. Now multiply through and then write each side of the result as a total derivative:

$$cR' - d \frac{R'}{R} = a \frac{F'}{F} - bF'$$

$$(cR - d \ln R)' = (a \ln F - bF)'$$

Functions that have the same derivative differ by a constant, so $cR - d \ln R = a \ln F - bF + K$. Rewriting,

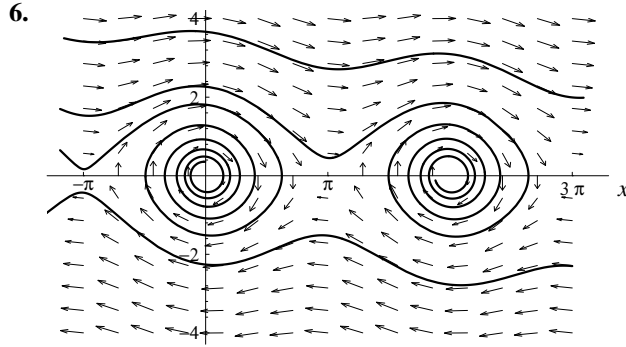
$$cR + bF - d \ln R - a \ln F = K.$$

Thus, if $R = R(t)$ and $F = F(t)$ form a solution to the Lotka-Volterra equations, then $cR + bF - d \ln R - a \ln F$ has a constant value. Therefore, it is a first integral of the system.

5. e. Step 1. Let $f(R) = cR - d \ln R$; f will be decreasing where $f' < 0$ and increasing where $f' > 0$. Since $f' = c - d/R$, f will be decreasing where $c - d/R < 0$, or $R < d/c$, and increasing where $R > d/c$.

Step 2. Now fix F : $F = F_0$, say. Evaluating the first integral, we get $cR + bF_0 - d \ln R - a \ln F_0 = K$, implying $f(R)$ is a constant (namely, $K + a \ln F_0 - bF_0$). Since $f(R)$ is decreasing on $0 < R < d/c$, there is at most one R in this interval for which $f(R)$ will equal this constant. Since $f(R)$ is increasing on $R > d/c$, we have a similar conclusion there. Thus, there are at most two positive values of R for which $f(R)$ has a given value.

Step 3. If $(R, F) = (R(t), F(t))$ were a spiral trajectory, there would be some horizontal line $F = F_0$ in the R, F -plane which it meets infinitely often. On the trajectory the first integral $cR + bF - d \ln R - a \ln F$ is constant, K say. At any point where the trajectory meets the line $F = F_0$, we can write $cR + bF_0 - d \ln R - a \ln F_0 = K$, or $f(R) = K + a \ln F_0 - bF_0$. By step 2, we know there are at most two values of R for which this is true. Thus, the trajectory cannot be a spiral. It can be a closed loop, though; a closed loop can meet a horizontal line in only two points.



The equilibrium points lie on the line $v = 0$, and satisfy $\sin x = 0$. Thus they are $(x, v) = (k\pi, 0)$. The phase portrait above shows that, when k is even, the equilibrium is a spiral attractor. When k is odd, the equilibrium is a saddle; this is most evident for the point $(-\pi, 0)$. Physically, there are only two equilibria: all *even* multiples of π represent the lower rest position of the pendulum, and all *odd* multiples represent the pendulum balanced upright.

All trajectories move gradually toward the x -axis; virtually all of them eventually spiral in to the rest position represented by $(2k\pi, 0)$. Physically, the pendulum “runs down”. (Two physically distinct trajectories find their way to the upright position represented by $((2k+1)\pi, 0)$).

7. Set $dE_1/dt = 0$ and solve for E_2 in terms of E_1 :

$$\begin{aligned} 0 &= \Lambda_1 - \mu_1 E_1 + \frac{a_1 E_1 E_2}{1 + b_1 E_1 E_2} \\ &= (\Lambda_1 - \mu_1 E_1)(1 + b_1 E_1 E_2) + a_1 E_1 E_2 \\ &= \Lambda_1 - \mu_1 E_1 + (\Lambda_1 - \mu_1 E_1 + a_1/b_1)b_1 E_1 E_2 \end{aligned}$$

Thus,

$$E_2 = \frac{\mu_1 E_1 - \Lambda_1}{b_1 E_1 (\Lambda_1 - \mu_1 E_1 + a_1/b_1)}.$$

All variables and parameters must be positive; in particular, $E_2 > 0$. This implies that the numerator and denominator are either both positive or both negative. If they are both negative, then

$$\Lambda_1 - \mu_1 E_1 + a_1/b_1 < 0.$$

But this is impossible, because $\Lambda_1 - \mu_1 E_1 > 0$ (this is the negative of the numerator) and a_1 and b_1 are positive.

So we need only consider what happens when numerator and denominator are both positive. We have $\mu_1 E_1 - \Lambda_1 > 0$ and $\Lambda_1 - \mu_1 E_1 + a_1/b_1 > 0$, implying $0 < \mu_1 E_1 - \Lambda_1 < a_1/b_1$ or, finally,

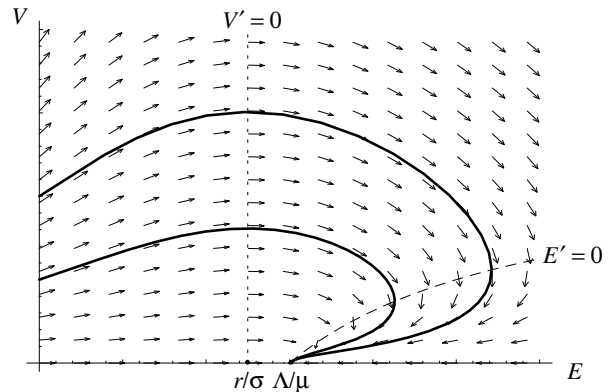
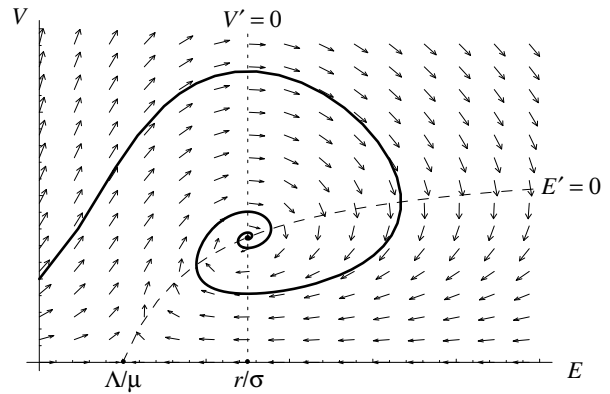
$$\frac{\Lambda_1}{\mu_1} < E_1 < \frac{a_1/b_1 + \Lambda_1}{\mu_1}.$$

This is an interval on the E_1 axis.

For the values of the parameters given in the text, $a_1/b_1 + \Lambda_1 = 1.0008 \times 10^7 \approx 10^7$, so we can write the inequalities as $8 < E_1 < 10^4$. In the text, the E_1 is consistently put on the vertical axis, so the graphs have to be read with extra care. Thus, $8 < E_1 < 10^4$ is an interval on the *vertical* axis.

8. a. In the absence of any virus, $V = 0$ so $dE/dt = \Lambda - \mu E = 0$. Thus we have an equilibrium ($dE/dt = 0$) at the point $E = \Lambda/\mu$. To see it is stable, we check what happens if E is above or below this value. If $E > \Lambda/\mu$, then $dE/dt < 0$ so E decreases (toward the equilibrium). If $E < \Lambda/\mu$, then $dE/dt > 0$ so E increases (toward the equilibrium). Since E always moves back toward the equilibrium, the equilibrium is stable *when the virus is absent*.

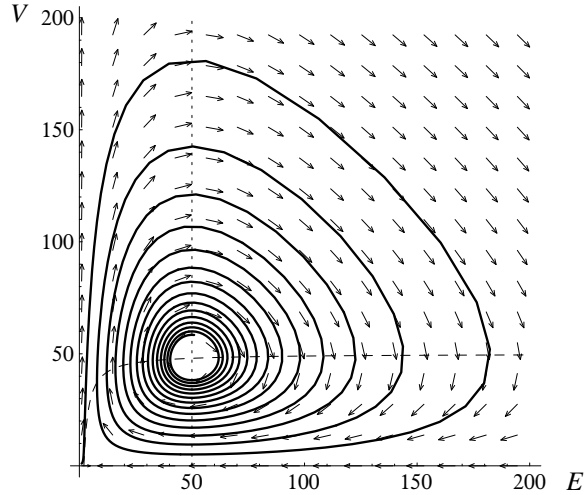
8. b. First determine where $E' = 0$ and $V' = 0$. Since $V' = (r - \sigma E)V$, $V' = 0$ on the E -axis and on the vertical line $E = r/\sigma$. $E' = 0$ on $V = (\mu E - \Lambda)/\epsilon E$, which is a hyperbola. It has positive slope and crosses the E -axis, entering the first quadrant, at the point $E = \Lambda/\mu$.



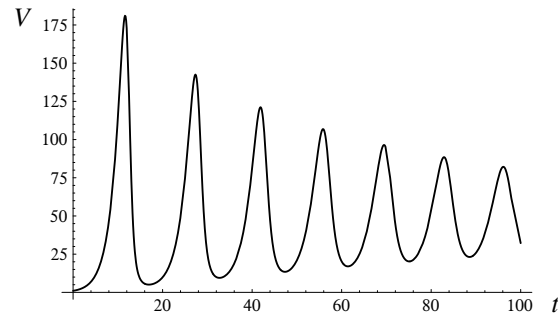
If the vertical line is to the right of entry point (so $\Lambda/\mu < r/\sigma$), then the vertical line intersects the hyperbola in the first quadrant, creating an equilibrium point there: $(r/\sigma, (\mu r - \sigma \Lambda)/\epsilon r)$. The first figure above illustrates this case. The equilibrium is a spiral attractor, and all initial conditions in the first quadrant are drawn to it. In the long term, both the virus and the effector cells persist.

If the vertical line lies on or to the left of the entry point ($\Lambda/\mu \geq r/\sigma$), there is no equilibrium point in the first quadrant. Trajectories in the first quadrant are drawn instead to the equilibrium at $E = \Lambda/\mu$ on the E -axis. This equilibrium is now an attractor. The second figure illustrates this case. In the long term, the virus disappears but the effector cells remain.

8. c. The trajectory is a spiral that approaches the equilibrium point at $(r/\sigma, (\mu r - \sigma\Lambda)/\epsilon r) = (50, 48)$, but it approaches equilibrium *very slowly*.



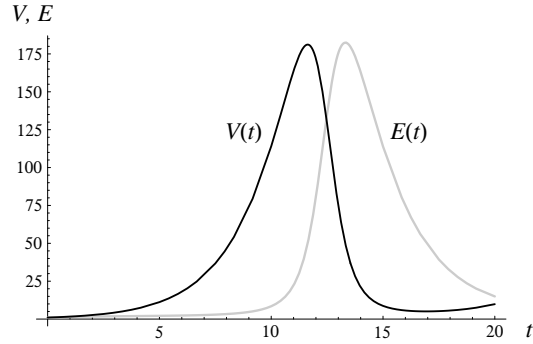
8. d. The following plot shows the virus $V(t)$ over a long time period. The graph has its first peak when $t \approx 12$ and its seventh when $t \approx 96$. Hence $V(t)$ goes through six cycles over about 84 time units, so $T \approx 14$. The cycles of $E(t)$ are the same length.



The following figure shows $V(t)$ and $E(t)$ (in gray) over the short time span $0 \leq t \leq 20$, which is slightly longer than a single cycle. We can read in to the graph the following sequence of events over the lifetime of an infected individual:

- 1) the virus infection gradually takes hold;
- 2) the effector cells increase in response;
- 3) the virus fades away;
- 4) the effector cells return to low levels.

It appears that the virus never returns.



8.2 Local Behavior of Dynamical Systems

1. We follow the steps outlined on pages 487–488 of the text.

1. a. (From 8.1, Exercise 2) There are four equilibrium points; $(B, C) = (131.25, 62.5)$ is the most complicated, so we do that one in detail. First, shift coordinates to the point: let $b = B - 131.25$ and $c = C - 62.5$.

Next, write the differential equations in terms of the new variables. The first is

$$\begin{aligned} b' &= .1[1 - .01(b + 131.25) + .005(c + 62.5)](b + 131.25) \\ &= 13.125(-.01b + .005c) + O(2) \\ &= -.13125b + .065625c + O(2) \end{aligned}$$

We use here the notation “ $O(2)$ ” to indicate *terms of order 2 and higher*—in other words, everything above the linear terms. (This notation is introduced in the text in chapter 9.) The second is

$$\begin{aligned} c' &= .03[1 + .04(b + 131.25) - .1(c + 62.5)](c + 62.5) \\ &= 1.875(.04b - .1c) + O(2) \\ &= .075b - .1875c + O(2) \end{aligned}$$

Thus, the local linearization at $(131.25, 62.5)$ has the form

$$\begin{aligned} b' &= -.13125b + .065625c \\ c' &= .075b - .1875c \end{aligned}$$

Here are the other equilibrium points:

$$\begin{aligned} (0, 0) \quad b' &= .1b \\ &c' = .03c \\ (100, 0) \quad b' &= -.1b + .05c \\ &c' = .15c \\ (0, 10) \quad b' &= .105b \\ &c' = .012b - .03c \end{aligned}$$

1. b. (Exercise 3) We use r and s for the shifted coordinates. Thus, for example, at the equilibrium $(x, y) = (200, 0)$ we set $r = x - 200$, $s = y$.

$$\begin{aligned} (0, 0) \quad r' &= .15r \\ s' &= .03s \\ (200, 0) \quad r' &= -.15r - .3s \\ s' &= .006s \\ (0, 200) \quad r' &= -.15r \\ s' &= -.024r - .030s \end{aligned}$$

1. c. (Exercise 4) There is a slight change in the form of the linearization at $(200, 0)$, but it is enough to convert what was a saddle in the previous part into an attractor. There is also a slight change in the linearization at the attractor $(0, 200)$. The new equilibrium $(1000/7, 200/7)$ is messy, but the computations will go through if you keep the fractions as fractions instead of converting them to decimal approximations.

$$\begin{aligned} (0, 0) \quad r' &= .15r \\ s' &= .03s \\ (200, 0) \quad r' &= -.15r - .3s \\ s' &= -.006s \\ (0, 200) \quad r' &= -.15r \\ s' &= -.036r - .030s \\ (1000/7, 200/7) \quad r' &= -\frac{3}{28}r - \frac{6}{28}s \\ s' &= -\frac{36}{7000}r - \frac{30}{7000}s \end{aligned}$$

2. a. $R' = 0$ implies $R(a - bF) = 0$ and thus $F = a/b$. $F' = 0$ implies $F(cR - d) = 0$ and thus $R = d/c$. Therefore, $(d/c, a/b)$ is an equilibrium point.

2. b. Let $r = R - d/c$, $f = F - a/b$. Then the differential equation $R' = aR - bRF$ becomes

$$\begin{aligned} r' &= a(r + d/c) - b(r + d/c)(f + a/b) \\ &= ar + ad/c - b(ad/bc + ar/b + df/c + O(2)) \\ &= ar + ad/c - ad/c - ar - bdf/c + O(2) \\ &= -\frac{bd}{c}f + O(2) \end{aligned}$$

In a similar way, $F' = cRF - dF$ becomes

$$f' = \frac{ac}{b}r + O(2)$$

The linearization of the Lotka-Volterra equations at $(d/c, a/b)$ is therefore

$$r' = -\frac{bd}{c}f \quad f' = \frac{ac}{b}r$$

2. c. In the linearization, the rate of change of each variable depends only on the other variable. Furthermore, in each quadrant (with respect to the *new* variables r, f), vectors point toward the next quadrant, so trajectories go in loops (which may or may not close) around the r, f origin. Hence, each of the variables r and f *oscillates*.

2. d. Suppose $(r(t), f(t))$ is a trajectory of the linearized system, and $E(t) = \alpha r^2(t) + \beta f^2(t)$. If E is to be a constant, then its derivative with respect to t must be zero:

$$\begin{aligned} E' &= 2\alpha rr' + 2\beta ff' \\ &= -2\alpha r \frac{bd}{c}f + 2\beta f \frac{ac}{b}r \\ &= 2rf \left(\beta \frac{ac}{b} - \alpha \frac{bd}{c} \right) \end{aligned}$$

Any choices for α and β that make the expression in parentheses equal zero will do; for example,

$$(\alpha, \beta) = \left(\frac{ac}{b}, \frac{bd}{c} \right) \quad \text{or} \quad \left(\frac{c}{bd}, \frac{b}{ac} \right).$$

3. In this problem, pairs of parameters described by the same letter (but with different subscripts) have the same value. We'll write each parameter without a subscript; thus $\Lambda = \Lambda_1 = \Lambda_2$, and so forth. Then the system is

$$\begin{aligned} \frac{dE_1}{dt} &= \Lambda - \mu E_1 + \frac{aE_1E_2}{1 + bE_1E_2} = f_1(E_1, E_2) \\ \frac{dE_2}{dt} &= \Lambda - \mu E_2 + \frac{aE_1E_2}{1 + bE_1E_2} = f_2(E_1, E_2). \end{aligned}$$

To linearize this system at the equilibrium point $P_1 = (P, P) = (8.7689, 8.7689)$, we introduce the new variables $r = E_1 - P$, $s = E_2 - P$. As explained on page 489, the linearized system is

$$\begin{aligned} r' &= \frac{\partial f_1}{\partial E_1}r + \frac{\partial f_1}{\partial E_2}s \\ s' &= \frac{\partial f_2}{\partial E_1}r + \frac{\partial f_2}{\partial E_2}s, \end{aligned}$$

where the partial derivatives are evaluated at $(E_1, E_2) = (P, P)$. One computes that

$$\begin{aligned} \frac{\partial f_1}{\partial E_1} &= \frac{\partial}{\partial E_1} \left(\Lambda - \mu E_1 + \frac{aE_1E_2}{1 + bE_1E_2} \right) \\ &= -\mu + \frac{(1 + bE_1E_2)(aE_2) - (bE_2)(aE_1E_2)}{(1 + bE_1E_2)^2} \\ &= -\mu + \frac{aE_2}{(1 + bE_1E_2)^2} = -\mu + \frac{aP}{(1 + bP^2)^2}, \end{aligned}$$

where the final equality is the result of evaluating at $(E_1, E_2) = (P, P)$. The other partial derivatives are equally easy, with the result that the linearized system is

$$\begin{aligned} r' &= \left(\frac{aP}{(1+bP^2)^2} - \mu \right) r + \frac{aP}{(1+bP^2)^2} s \\ s' &= \frac{aP}{(1+bP^2)^2} r + \left(\frac{aP}{(1+bP^2)^2} - \mu \right) s \end{aligned}$$

or, in terms of the parameter values $\mu = 1000$, $a = 10$, and $b = 10^{-6}$,

$$\begin{aligned} r' &= -912.324r + 87.6755s \\ s' &= 87.6755r - 912.324s. \end{aligned}$$

4. Write the second Anderson–May model in the form

$$\begin{aligned} E' &= \Lambda - \mu E + \varepsilon V E = f_1(E, V) \\ V' &= rV - \sigma V E = f_2(E, V). \end{aligned}$$

The analysis of part (b) of the previous Exercise 8 (see page 483) shows that the system has two equilibrium points, one at $(E, V) = (\Lambda/\mu, 0)$ and the other at $(E, V) = (r/\sigma, (\mu r - \sigma\Lambda)/(\varepsilon r))$. Each equilibrium point has local coordinates e and v , and as explained on page 489, the local linearization is given by

$$\begin{aligned} E' &= \frac{\partial f_1}{\partial E} e + \frac{\partial f_1}{\partial V} v = (-\mu + \varepsilon V)e + \varepsilon E v \\ V' &= \frac{\partial f_2}{\partial E} e + \frac{\partial f_2}{\partial V} v = -\sigma V e + (r - \sigma E)v, \end{aligned}$$

where the partial derivatives are evaluated at the equilibrium point.

For the equilibrium point $(E, V) = (\Lambda/\mu, 0)$, this gives the local linearization

$$\begin{aligned} e' &= -\mu e + \frac{\varepsilon\Lambda}{\mu} v = -.5e + .02v \\ v' &= \left(\frac{\mu r - \sigma\Lambda}{\mu} \right) v = .48v, \end{aligned}$$

where the final equality on each line uses the parameter values $\Lambda = 1$, $\mu = r = .5$ and $\varepsilon = \sigma = .01$ from part (c) of the previous exercise 8 (see page 483).

At the equilibrium $(E, V) = (r/\sigma, (\mu r - \sigma\Lambda)/\varepsilon r)$, we get the local linearization

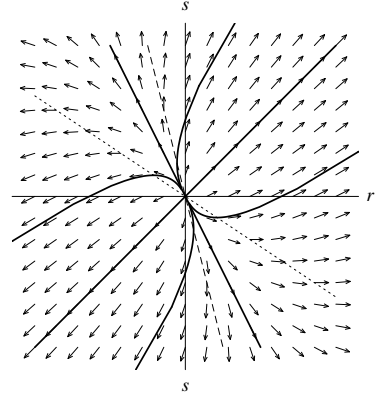
$$\begin{aligned} e' &= -\frac{\sigma\Lambda}{r} e + \frac{\varepsilon r}{\sigma} v = -.02e + .5v \\ v' &= -\sigma \frac{\mu r - \sigma\Lambda}{\varepsilon r} e = -.12e, \end{aligned}$$

where again the final equality on each line uses the parameter values given in part (c) of the previous exercise 8 (see page 483).

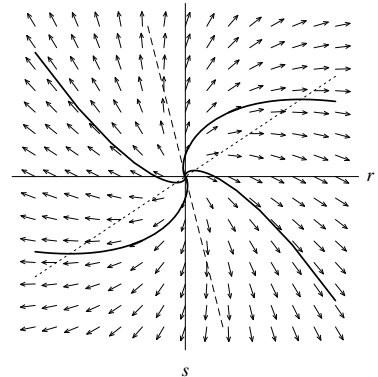
8.3 A Taxonomy of Equilibrium Points

1. In each of the following, $s' = 0$ is a dotted line, $r' = 0$ is a dashed line, and straight line trajectories are shown as solid lines when they exist.

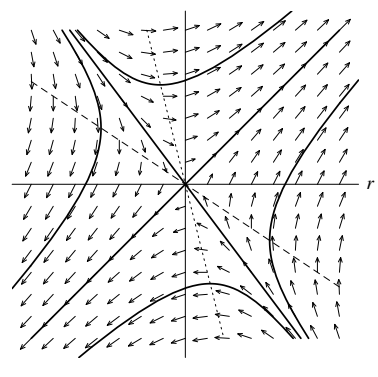
1. a. Node repellor. Straight line trajectories have slopes $m = -2, 1$.
 $r' = 0$ on $s = -4r$
 $s' = 0$ on $s = -\frac{2}{3}r$



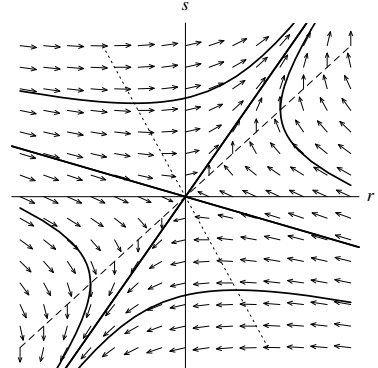
1. b. Spiral repellor. No straight line trajectories.
 $r' = 0$ on $s = -4r$
 $s' = 0$ on $s = \frac{2}{3}r$



1. c. Saddle. Straight line trajectories have slopes $m = -\frac{4}{3}, 1$.
 $r' = 0$ on $s = -\frac{2}{3}r$
 $s' = 0$ on $s = -4r$



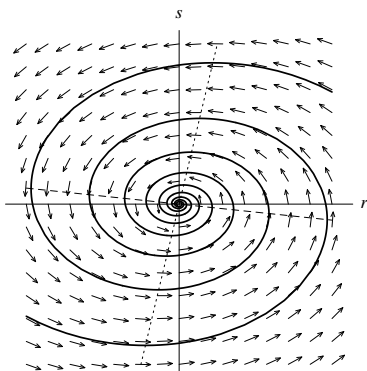
1. d. Saddle. Straight line trajectories have slopes $m = (5 \pm \sqrt{57})/8$.
 $r' = 0$ on $s = r$
 $s' = 0$ on $s = -2r$



- 1. e.** Spiral attractor.
No straight line trajectories.

$$r' = 0 \text{ on } s = -.1r$$

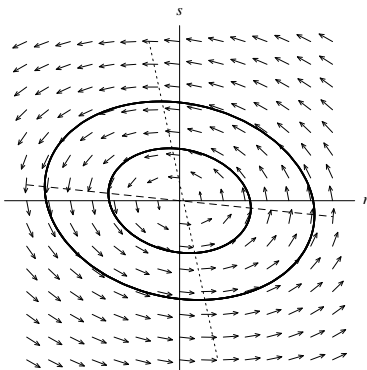
$$s' = 0 \text{ on } s = 4r$$



- 1. f.** Center.
No straight line trajectories.

$$r' = 0 \text{ on } s = -.1r$$

$$s' = 0 \text{ on } s = -5r$$



- 1. g.** *Ad libitum*; the answer depends on choices made by the student.

2. The trajectories of the new system are the same curves as the original; the only difference is that the state evolves along a trajectory four times as fast as it did originally.

3. The trajectories of the new system are the same curves as the original; however, the arrow (that shows the direction in which the state evolves along the trajectory) reverses direction.

4. a. Since $b \neq 0$, we can write the equation for slopes in the form $bm^2 + (a-d)m - c = 0$. The quadratic equation gives the solutions

$$m = \frac{-(a-d) \pm \sqrt{(a-d)^2 + 4bc}}{2b}.$$

4. b. Section 8.2, exercise 1a:

- $b' = .1b, c' = .03c$. Lines are $c = 0$ and $b = 0$; flow is outward on both.
- $b' = -.1b + .05c, c' = .15c$. Lines are $c = 0$ and $c = 5b$; flow is inward on the first, outward on the second.
- $b' = .105b, c' = .012b - .03c$. Lines are $b = 0$ and $c = \frac{4}{45}b$; flow is outward on the first, inward on the second.
- $b' = -.13125b + .065625c, c' = .075b - .1875c$. Lines are $c = -.006806b, c = .003145b$; flow is inward on both lines.

Section 8.2, exercise 1b:

- $r' = .15r, s' = .03s$. Lines are $s = 0$ and $r = 0$; flow is outward on both.

- $r' = -.15r - .3s, s' = .006s$. Lines are $s = -.52r$ and $s = 0$; flow is outward on first and inward on second.

- $r' = -.15r, s' = -.024r - .030s$. Lines are $s = .2r$ and $r = 0$; flow is inward on both.

Section 8.2, exercise 1c:

- $r' = .15r, s' = .03s$. Same as 1b.

- $r' = -.15r - .3s, s' = -.006s$. Lines are $s = -.48r$ and $s = 0$; flow is inward on both.

- $r' = -.15r, s' = -.036r - .030s$. Lines are $s = .3r$ and $r = 0$; flow is inward on both.

- $r' = -\frac{3}{28}r - \frac{6}{28}s, s' = -\frac{36}{7000}r - \frac{30}{7000}s$. This is the example in the text, pages 493–494, but we get slightly different results because we start with exact values for a, b, c , and d . The lines are $s = -0.525657r$ and $s = 0.0456571r$; flow is outward on the first, inward on the second.

Section 8.2, exercise 3:

- $e'_1 = -912.325e_1 + 87.6752e_2, e'_2 = 87.6752e_1 - 912.325e_2$. Lines are $e_2 = \pm e_1$; flow is inward on both.

Section 8.2, exercise 4:

- $e' = -.5e + .02v, v' = .48v$. Lines are $v = 49e$ and $v = 0$; flow is outward on the first, inward on the second.

- $e' = -.02e + .5v, v' = -.48v$. Lines are $v = -.92e$ and $v = 0$; flow is inward on both.

4. c. The system will have straight line trajectories when the quadratic equation $bm^2 + (a-d)m - c = 0$ in part (a) has real roots; this happens precisely when the discriminant is non-negative: $(a-d)^2 + 4bc \geq 0$.

5. The equation $bm^2 + (a-d)m - c = 0$ for straight line trajectories must reduce to the equation $m^2 - 1 = 0$ whose roots are ± 1 . Thus $a = d$ and $b = c$. If $b = c = 0$, then every straight line through the origin is a trajectory.

6. The quadratic equation for m must have a repeated root. This happens when the discriminant is 0; in this case, $(a-d)^2 + 4bc = 0$. Students can choose possibilities *ad libitum*, so long as they satisfy this condition. A typical simple case is $x' = ax + y, y' = ay$.

7. a. Linearized system

$$r' = 0, s' = -s$$

Every point on the r -axis is an equilibrium.

Trajectories flow vertically toward the r -axis.

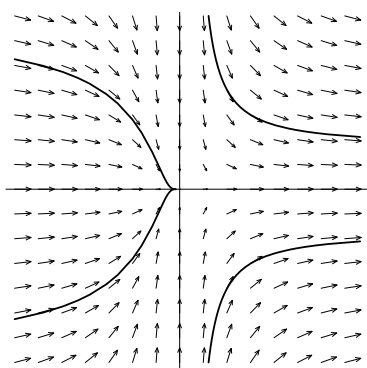
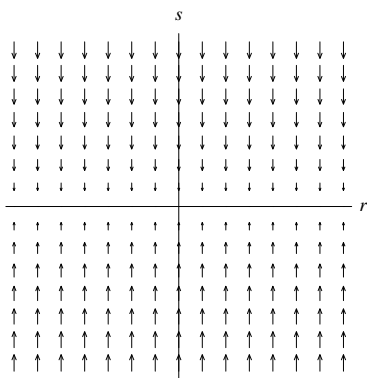
$$r' = 0 \text{ everywhere} \\ s' = 0 \text{ on } r\text{-axis.}$$

Original system

$$r' = r^2, s' = -s$$

The right half-plane looks like a saddle; the left half like a node attractor. All trajectories approach the r -axis; as in the linearized system.

$$r' = 0 \text{ on } s\text{-axis} \\ s' = 0 \text{ on } r\text{-axis.}$$

**7. b. Linearized system**

$$r' = 0, s' = r$$

Every point on the s -axis is an equilibrium.

Trajectories flow vertically upwards to the right of the s -axis, downwards to the left.

Trajectories further from the s -axis flow more rapidly.

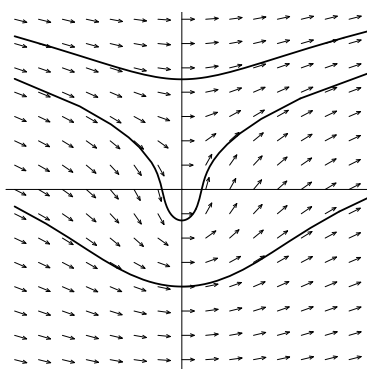
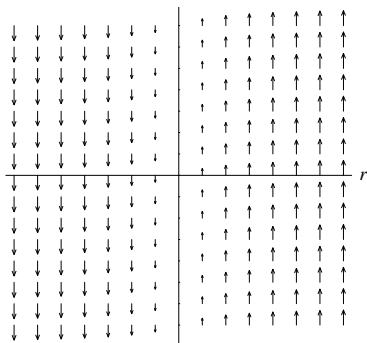
$$r' = 0 \text{ everywhere; } s' = 0 \text{ on } s\text{-axis.}$$

Original system

$$r' = r^2 + s^2, s' = r$$

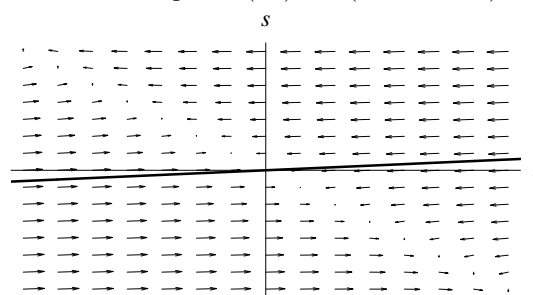
The flow is generally left-to-right, but near the origin trajectories flow down to the left and up to the right, as in the linearized system.

$$r' = 0 \text{ nowhere} \\ s' = 0 \text{ on } s\text{-axis.}$$



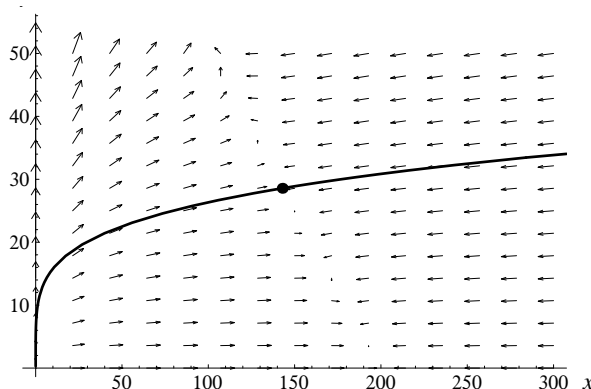
8. a. This part concerns the linear system $r' = -.1071r - .2143s$, $s' = -.0051r - .0043s$. The instructions are to draw trajectories of the negative of this system whose initial points are on the line $s = .0454r$ and lie near the origin,

on either side of it. The trajectories in the figure below are chosen with initial points $(r, s) = \pm(.01, .000454)$.



8. b. The system is $x' = .15(1 - .005x - .010y)x$, $y' = .03(1 - .006x - .005y)y$, and the equilibrium point is at $(x, y) = (1000/7, 200/7)$. The boundary curve in the figure below was produced using the initial points

$$(x, y) = \left(\frac{1000}{7} \pm 1, \frac{200}{7} \pm .0454 \right).$$

**Distance from the Origin**

9. a. Apply the chain rule to $D(t) = r(t)^2 + s(t)^2$ to get $D' = 2rr' + 2ss'$. The text does the rest!

9. b. Let $s = mr$ in the equation $D' = 2[ar^2 + (b+c)rs + ds^2]$. The result is $D' = 2[ar^2 + (b+c)mr^2 + dm^2r^2] = 2r^2[a + (b+c)m + dm^2]$. Since r is not identically zero, $D' = 0$ if and only if $a + (b+c)m + dm^2 = 0$.

9. c. The quadratic formula gives the solutions to

$$a + (b+c)m + dm^2 = 0$$

in the form presented in the exercise.

9. d. The quadratic equation has no real roots when $(b+c)^2 - 4ad < 0$; D' is never zero, so D is either always increasing or always decreasing on a trajectory.

9. e. The equations of the two lines of “vertices” of trajectories are $s = -1.0267r$, $s = -49.9965r$.

10. We do all parts of Exercise 1 from this section. If there are no lines along which trajectories make their closest approach, we say *No*; otherwise, we give the slopes m of the lines.

10. a. $(b+c)^2 - 4ad = 3^2 - 4 \cdot 12 < 0$. No.

10. b. $(b+c)^2 - 4ad = (-1)^2 - 4 \cdot 12 < 0$. No.

10. c. $(b+c)^2 - 4ad = 7^2 - 4 \cdot 2 = 41$. $m = (-7 \pm \sqrt{41})/2$.

10. d. $(b+c)^2 - 4ad = 6^2 + 16 = 52$. $m = -3 \pm \sqrt{13}$.

10. e. $(b+c)^2 - 4ad = (-2)^2 - 4(.2) = 3.2$. $m = -4 \pm \sqrt{14.8}$.

10. f. $(b+c)^2 - 4ad = (-2)^2 + 4(.16) = 4.64$. $m = (10 \pm \sqrt{116})/4$.

11. Assume the system has the form $r' = ar + bs$, $s' = cr + ds$. Any example a student constructs must meet the following conditions.

11. a. Spirals are systems with no straight line trajectories. The condition for this (see exercise 4c) is $(a-d)^2 + 4bc < 0$. The second condition, that trajectories move steadily either towards or away from the origin, is that $(b+c)^2 - 4ad < 0$ (exercise 9c). The condition that trajectories move *away* from the origin is the condition that $D' > 0$. But $D' = 2r^2[dm^2 + (b+c)m + a]$. Since we want $D' > 0$, the graph of D' as a function of m must open upward: $d > 0$. (Since $d > 0$ and $(b+c)^2 < 4ad$ as well, it follows that $a > 0$.)

11. b. We now want $(b+c)^2 - 4ad > 0$. The other conditions continue to hold: $(a-d)^2 + 4bc < 0$, $a > 0$, $d > 0$.

12. The lines are perpendicular if their slopes are negative reciprocals of each other. Taking the slopes from exercise 9c, we find the negative reciprocal condition is

$$\frac{-(b+c) + \sqrt{(b+c)^2 - 4ad}}{2d} = \frac{-2d}{-(b+c) - \sqrt{(b+c)^2 - 4ad}}.$$

Doing the algebra, we get

$$\begin{aligned}(b+c)^2 - [(b+c)^2 - 4ad] &= -4d^2 \\ 4ad &= -4d^2 \\ a &= -d\end{aligned}$$

There is another way to do this. The product of the roots of the equation $dm^2 + (b+c)m + a = 0$ is a/d . Since the roots must be negative reciprocals, their product is -1 , so $a/d = -1$, implying $a = -d$.

13. No trajectories are straight lines (because they are all closed loops). Hence $(a-d)^2 + 4bc < 0$. If the loops are “flattened”, then by exercise 12 the condition $a = -d$ must also be satisfied.

8.4 Limit Cycles

1. At the equilibrium point, $x' = y' = 0$. From $y' = 0$ we see $y = fx$. Substituting $y = fx$ into $x' = 0$ (and factoring out x), we get

$$a\left(1 - \frac{x}{b}\right) - \frac{cfx}{x+d} = 0$$

which reduces to $ax^2 + (ad + bcf - ab)x - abd = 0$, or $.6x^2 + 4.6x - 6 = 0$. The positive root of this is $x = (-4.6 + \sqrt{35.56})/1.2 = 1.13602$. Thus the equilibrium point is at $(1.13602, 2.27204)$.

To show it is a repeller, linearize the system at that point. Let $r = x - 1.13602$, $s = y - 2.27204$. Then $x' = .6x(1 - .1x) - .5xy/(1+x)$ becomes

$$\begin{aligned}r' &= .6(r + 1.13602)(1 - .1(r + 1.13602)) \\ &\quad - \frac{.5(r + 1.13602)(s + 2.27204)}{2.13602 + r}\end{aligned}$$

The tricky part of the linearization is the denominator. We use the fact $1/(1+A) = 1 - A + O(2)$ (see the solution to exercise 3, section 8.2) to write the second term as

$$\begin{aligned}&\frac{-.5}{2.13602} \frac{(r + 1.13602)(s + 2.27204)}{1 + r/2.13602} = \\ &\frac{-.5(r + 1.13602)(s + 2.27204)(1 - r/2.13602 + O(2))}{2.13602}\end{aligned}$$

With a lot of algebra, this reduces to

$$-.60418 - .248986r - .26592s + O(2).$$

The first term simplifies to $.60418 + .463678r + O(2)$; putting these together, we get

$$r' = .214692r - .26592s + O(2).$$

Next, $y' = .1y(1 - .5y/x)$. This becomes

$$\begin{aligned}s' &= .1(s + 2.27204) \left(1 - \frac{.5(s + 2.27204)}{r + 1.13602}\right) \\ &= .1(s + 2.27204) - \frac{.05}{1.13602} \frac{(s + 2.27204)^2}{1 + r/1.13602}\end{aligned}$$

The second term reduces to

$$\begin{aligned}&(-.227204 - .2s + O(2))(1 - r/1.13602 + O(2)) \\ &= -.227204 + .2r - .2s + O(2)\end{aligned}$$

This gives us $s' = .2r - .1s + O(2)$. The linearization is therefore

$$r' = .214692r - .26592s \quad s' = .2r - .1s.$$

The criteria of section 8.3 tell us that the linear equilibrium is a spiral (because $(a-d)^2 + 4bc = -.113705 < 0$). Furthermore, the spirals are “flattened” because $(b+c)^2 - 4ad = .090222 > 0$. However, no criterion given there shows conclusively that the equilibrium is a repeller. It is, though; this can be seen by plotting trajectories.

2. *Ad libitum*. One general note: when the equilibrium is at the moment of transition from a repeller to an attractor, the linearization will become “degenerate”, in the sense that the two differential equations will become multiples of each other.

8.5 Beyond the Plane: Three-dimensional Systems

1. The x' equation is a standard Lotka-Volterra equation for a predator x in a predator-prey system with prey y . In the y' equation, y plays the role of prey in relation to x (this is reflected in the term $-.001xy$) but the role of predator in relation to z (in the term $+.000001yz$). In the absence of plants z , y disappears (in the term $-.05y$). In the z' equation, the terms $z - .00001z^2$ indicate growth with a carrying capacity (due, for example, to overcrowding) while the term $-.0001yz$ reflects predation by y .

1. a. We can write the equations that determine the equilibrium points this way:

$$\begin{aligned} x(-2000 + y) &= 0 \\ y(-50000 - 1000x + z) &= 0 \\ z(100000 - z - 10y) &= 0 \end{aligned}$$

The following table describes a “decision tree” for finding all solutions by stepping through the equations one at a time.

first step	second step	final step
$x = 0$	$y = 0$	$z = 0$ $z = 100000$
	$z = 50000$	$y = 5000$
$y = 2000$	$z = 80000$	$x = 30$
	$z = 0$	$z = -50$

Here are the five points:

$A = (0, 0, 0)$	nothing
$B = (0, 0, 100000)$	plants only
$C = (0, 5000, 50000)$	plants and grazers
$D = (30, 2000, 80000)$	all species
$E = (-50, 2000, 0)$	unreal

1. b. Let $u = x - 30$, $v = y - 2000$, $w = z - 80000$. The differential equations become

$$\begin{aligned} u' &= -.2(u + 30) + .0001(u + 30)(v + 2000) \\ &= -.2u - 6 + 6 + .2u + .003v + O(2) \\ v' &= -.05(v + 2000) - .001(u + 30)(v + 2000) \\ &\quad + .000001(v + 2000)(w + 80000) \\ &= -.05v - 100 - 60 - 2u - .03v \\ &\quad + 160 + .08v + .002w + O(2) \\ &= -2u + .002w + O(2) \\ w' &= w + 80000 - .00001(w + 80000)^2 \\ &\quad - .0001(v + 2000)(w + 80000) \\ &= w + 80000 - 64000 - 1.6w \\ &\quad - 16000 - 8v - .2w + O(2) \\ &= -8v - .8w + O(2) \end{aligned}$$

The linearized versions of these agree with those given in the exercise.

Note that the rest of the discussion concerns the equilibrium of the *linearized* system—not the original one. However, theorems exist to guarantee that the original equilibrium shares the behavior of the linearized one.

Let $D = 8 \cdot 10^6 u^2 + 12000 v^2 + 3w^2$; then

$$\begin{aligned} D' &= 16 \cdot 10^6 uu' + 24000 vv' + 6ww' \\ &= 16 \cdot 10^6 u(.003v) + 24000 v(-2u + .002w) \\ &\quad + 6w(-8v - .8w) \\ &= 48000 uv - 48000 uv + 48vw - 48vw - 4.8w^2 \\ &= -4.8w^2 \end{aligned}$$

Since $D' \leq 0$ everywhere, D is a non-increasing function. In fact, unless $w = 0$, $D' < 0$ so D is strictly decreasing. The only way a trajectory could fail to get steadily closer to the equilibrium is if it remained in the plane $w = 0$. But if $w = 0$, then $w' = -8v$, so the w -component of the trajectory will change (and hence become different from 0) if $v \neq 0$. The only way this could fail to happen is if the trajectory remained on the line $w = v = 0$. But then $v' = -2u$, so the v -component of the trajectory will change

(and hence become different from 0) if $u \neq 0$. The only way this could fail to happen is if the trajectory remained at the point $w = v = u = 0$; but then $D = 0$. Thus, every trajectory not already at equilibrium approaches equilibrium; the equilibrium is an attractor.

2. a. In the x' equation, the first two terms describe positive growth with a carrying capacity. The remaining terms are interactions: the first ($+.002xy$) reflects the positive effect of ants upon violets; the second ($-.1xz$) reflects the negative effect of mice upon violets.

In the y' equation, the first two terms describe positive growth with a carrying capacity. The remaining (interaction) term reflects the positive effect of violets upon ants.

In the z' equation, the first term indicates that the mice will die out without the violets; the second (interaction) term reflects the positive effect of violets upon mice.

2. b. The equations reduce to

$$x(1000 - x + 2y - 100z) = 0$$

$$y(1000 - 10y + x) = 0$$

$$z(-1000 + x) = 0$$

first step	second step	final step
$z = 0$	$y = 0$	$x = 0$
		$x = 1000$
	$x = 0$	$y = 100$
$x = 1000$	$x = 1500$	$y = 250$
	$y = 0$	$z = 0^*$
	$y = 200$	$z = 4$

Notice that there is a repeat, marked with a *. The five distinct points are:

$A = (0, 0, 0)$	nothing
$B = (1000, 0, 0)$	violets only
$C = (0, 100, 0)$	ants only
$D = (1500, 250, 0)$	violets and ants
$E = (1000, 200, 4)$	all species

2. c. • At $(0, 0, 0)$: $u' = u$, $v' = v$, $w' = -w$.

• At $(1000, 0, 0)$:

$$u' = -u + 2v - 100w, \quad v' = 2v, \quad w' = 0.$$

• At $(0, 100, 0)$:

$$u' = 1.2u, \quad v' = .1u - v, \quad w' = -w.$$

• At $(1500, 250, 0)$:

$$u' = -1.5u - 3v - 150w, \quad v' = .25u - 2.5v, \quad w' = .5w.$$

Here are the details for the last equilibrium. Let $u =$

$x - 1500$, $v = 250$, $w = z$. Then the x' equation becomes

$$\begin{aligned} u' &= u + 1500 - .001(u + 1500)^2 \\ &\quad + .002(u + 1500)(v + 250) - .1(u + 1500)w \\ &= u + 1500 - 2250 - 3u + 750 + .5u + 3v \\ &\quad - 150w + O(2) \\ &= -1.5u + 3v - 150w + O(2) \end{aligned}$$

The y' equation becomes

$$\begin{aligned} v' &= v + 250 - .01(v + 250)^2 \\ &\quad + .001(u + 1500)(v + 250) \\ &= v + 250 - 625 - 5v + 375 + .25u + 1.5v + O(2) \\ &= .25u - 2.5v + O(2) \end{aligned}$$

The z' equation becomes

$$\begin{aligned} w' &= -w + .0001(u + 1500)w \\ &= -w + 1.5w + O(2) \\ &= .5w + O(2) \end{aligned}$$

2. d. Let $R = u^2 + v^2 + 25000w^2$. Then

$$\begin{aligned} R' &= 2uu' + 2vv' + 50000ww' \\ &= 2u(-u + 2v + -100w) + 2v(.2u - 2v) \\ &\quad + 50000w(.004u) \\ &= -2u^2 + 4uv - 200uw + .4uv - 4v^2 + 200uw \\ &= -2u^2 + 4.4uv - 4v^2 \\ &= -2[(u - 1.1v)^2 + .79v^2] \end{aligned}$$

The last step involved “completing a square”. Since the bracketed expression is always positive (unless $u = v = 0$), R is decreasing. The only way a trajectory could fail to move closer to the equilibrium is if it stayed on the line $u = v = 0$. But on this line $u' = -100w$, so the u -component of the trajectory will change (and hence become different from 0). So the trajectory cannot remain on the line $u = v = 0$.

Note this demonstrates the stability of only the *linearized* system. As in the preceding exercise, additional theorems must be brought to bear to show the original system copies the behavior of its linearization.

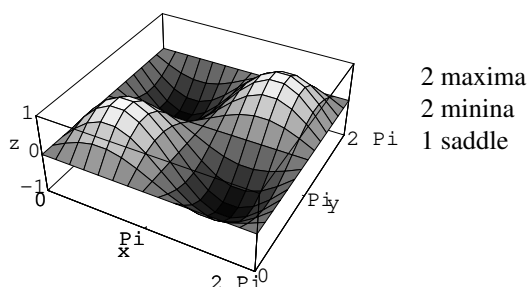
Solutions: Chapter 9

Functions of Several Variables

9.1 Graphs and Level Sets

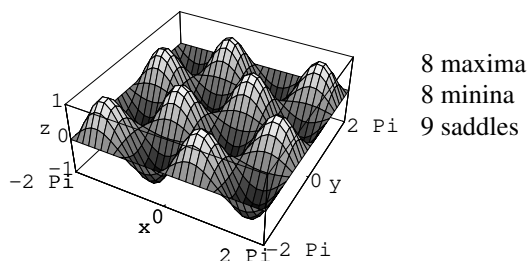
The graphs and contour plots here were produced by Mathematica 3 using the standard Mathematica shading scheme: low values are dark, high values are light. This is the opposite of what is used in the text itself.

1. a.



1. b. The saddle appears to be in the center of the domain, at $(x, y) = (\pi, \pi)$. The maxima and minima are at the “one quarter” points: the maxima at $(\pi/2, \pi/2)$ and $(3\pi/2, 3\pi/2)$; the minima at $(\pi/2, 3\pi/2)$ and $(3\pi/2, \pi/2)$.

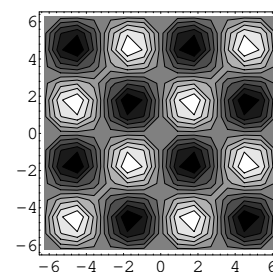
2. a.



2. b. The graph *does* look like an egg carton. The saddles are at the grid points where x and y are whole multiples of π , and the maxima and minima are evenly-spaced between them—alternating, like black and white squares on a checkerboard. (The next exercise shows the checkerboard.)

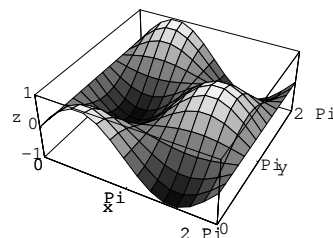
saddles	maxima	minima
$(-\pi, -\pi)$	$(-3\pi/2, -3\pi/2)$	$(-\pi/2, -3\pi/2)$
$(0, -\pi)$	$(\pi/2, -3\pi/2)$	$(3\pi/2, -3\pi/2)$
$(\pi, -\pi)$	$(-\pi/2, -\pi/2)$	$(-3\pi/2, -\pi/2)$
$(-\pi, 0)$	$(3\pi/2, -\pi/2)$	$(\pi/2, -\pi/2)$
$(0, 0)$	$(-3\pi/2, \pi/2)$	$(-\pi/2, \pi/2)$
$(\pi, 0)$	$(\pi/2, \pi/2)$	$(3\pi/2, \pi/2)$
$(-\pi, \pi)$	$(-\pi/2, 3\pi/2)$	$(-3\pi/2, 3\pi/2)$
$(0, \pi)$	$(3\pi/2, 3\pi/2)$	$(\pi/2, 3\pi/2)$
(π, π)		

3.



The number and locations of the maxima, minima, and saddles agree with what was found in the previous exercise. However, it is somewhat easier to confirm the coordinates of the points with the contour plot. (The irregularity of this plot is an artefact. It is due to using the default value of `PlotPoints` in Mathematica; a larger value would have made the contours appear smooth.)

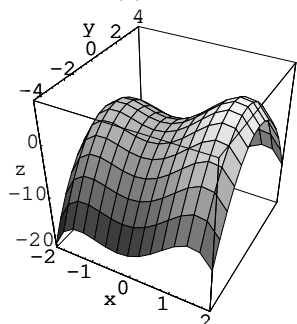
4.



The graph appears to have the same overall pattern as that in Exercise 1, but they differ by a shift: to make the graph

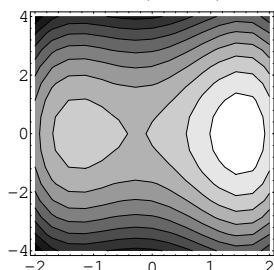
of Exercise 1 match this one, it is necessary to shift it by $-\pi/2$ in the direction of the y -axis. Since the function in this exercise is $\sin(x)\cos(y)$, while that in exercise 1 is $\sin(x)\sin(y)$, this makes sense, because a shift of $-\pi/2$ converts the graph of $\sin(y)$ into the graph of $\cos(y)$.

5.



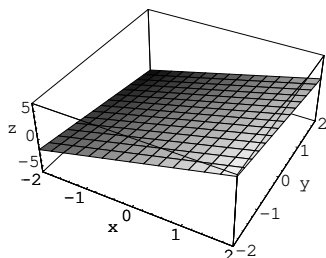
The graph shows that the minimum occurs at two corners of the domain: $(-2, \pm 4)$. The maxima and the saddle occur along the line $y = 0$, where the function reduces to $z = 2x + 4x^2 - x^4$. The maxima are near $(1.5, 0)$ and $(-1.3, 0)$; the saddle is near $(-0.3, 0)$.

6.



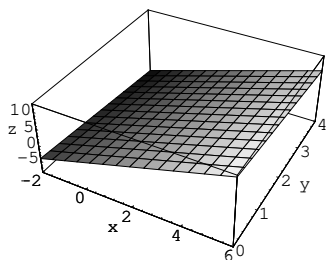
The contour plot shows local maxima and a saddle along the line $y = 0$, and it shows that that minimum occurs at the two left corners. The plot confirms the information extracted from the graph in the previous exercise.

7. a.



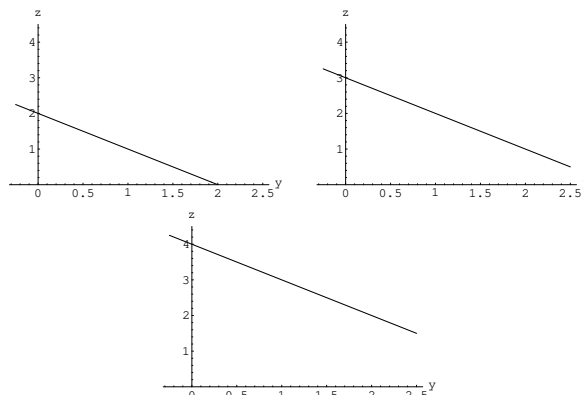
The graph is a tilted plane with a slope of 2 in the x -direction and a slope of -1 in the y -direction.

7. b.

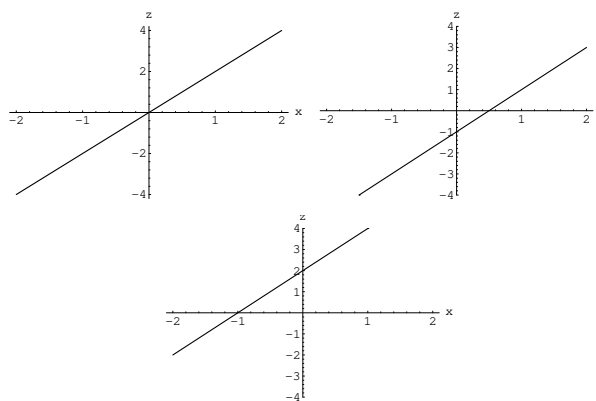


The graph is the same tilted plane as in the previous part; it is simply the part of the plane lying over a different domain.

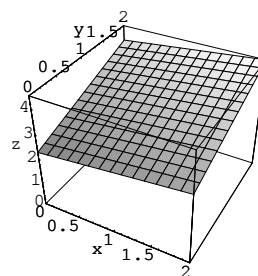
8. a. We get a slice in the y -direction by setting x equal to a constant. If $x = c$ then $z = 2c - y$. All slices are straight lines with slope -1 ; they differ only in their z -intercept. (The following figures illustrate this with $c = 1, 1.5$, and 2 .)



8. b. This time, $y = c$, yielding $z = 2x - c$. All slices are straight lines with slope $+2$; they differ only in their z -intercept. We chose $c = 0, 1$, and -2 in the following figures.



9. a. The following graph is plotted over the domain $0 \leq x, y \leq 2$. The z -intercept is $z = 2.3$, and is obtained by setting $x = y = 0$.

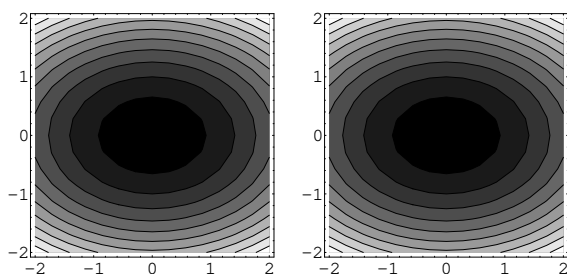


9. b. A vertical slice in the y -direction is obtained by setting x equal to a constant. If $x = c$ then $z = .8y + 2.3 + .3c$. This is a straight line with slope .8.

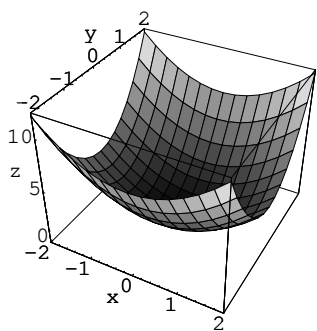
A vertical slice in the x -direction is obtained by setting y equal to a constant. If $y = c$ then $z = .3x + 2.3 + .8c$. This is a straight line with slope .3.

10. Let $z = px + qy + r$. A vertical slice of this graph in the y -direction is obtained by setting x equal to a constant: $x = c$. Then $z = qy + (pc + r)$, which is a straight line of slope q . A vertical slice in the x -direction is obtained by setting $y = c$. Then $z = px + (r + qc)$, which is a straight line of slope p .

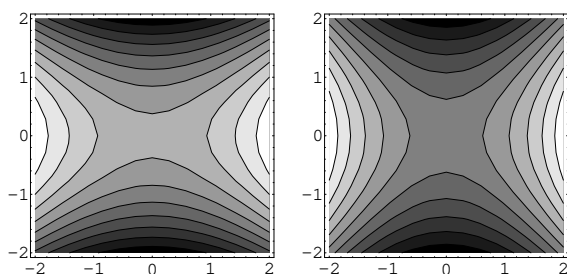
11. a. The contours of $z = x^2 + 2y^2$ are ellipses, while those of $z = x^2 + y^2$ are circles.



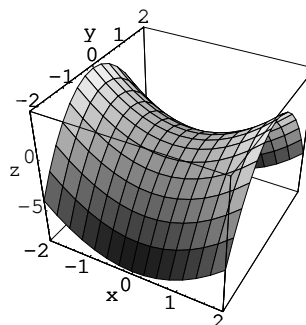
11. b. The graph of $z = x^2 + 2y^2$ is an elliptical bowl that rises more steeply in the y -direction than in the x -direction.



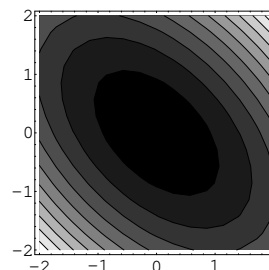
12. a. The contours of $z = x^2 - 2y^2$ and $z = x^2 - y^2$ both consist of hyperbolas. The asymptotes of the first are $y = \pm x/\sqrt{2}$; those of the second are $y = \pm x$.



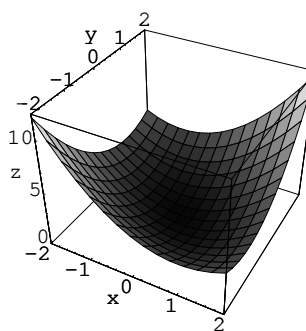
12. b. The graph of $z = x^2 - 2y^2$ should be a saddle because its contours show the characteristic “minimax” pattern of higher values at two sides and lower values at the other two.



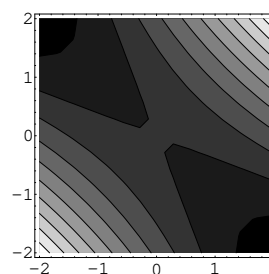
13. a.



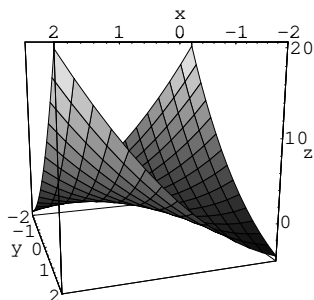
13. b. Since the contours are ovals, the graph should be a bowl. Moreover, the bowl should rise more steeply in the direction of the line $y = x$, and slower in the direction of the line $y = -x$. This is borne out by the graph.



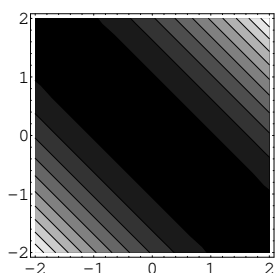
14. a.



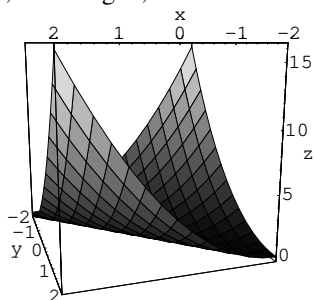
14. b. The contour plot has the characteristic shape of a minimax, so the graph should be a saddle. Furthermore, the graph should descend along the line $y = -x$ (because the contour plot shows the lowest values occur there) and ascend along the line $y = x$. This is borne out by the graph.



15. a.



15. b. According to the contour plot, the z -value is the same at all points along the line $y = -x$ (while the value of z increases as the (x, y) point moves away from this line). Therefore the graph is neither a bowl or a saddle. Instead, it is a “gutter”, or “trough”; this is borne out by the graph itself.

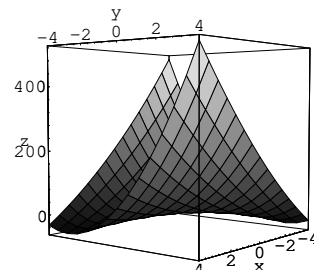
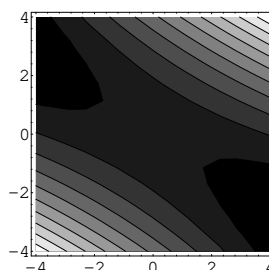


16. What happens depends on whether $f(x, y; p)$ is essentially the *sum* or the *difference* of two squares—and this can be determined by completing the square:

$$\begin{aligned} x^2 + pxy + 4y^2 &= x^2 + pxy + \frac{p^2}{4}y^2 - \frac{p^2}{4}y^2 + 4y^2 \\ &= \left(x + \frac{p}{2}y\right)^2 + \left(4 - \frac{p^2}{4}\right)y^2 \end{aligned}$$

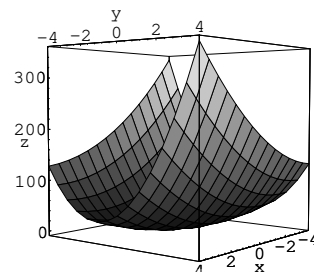
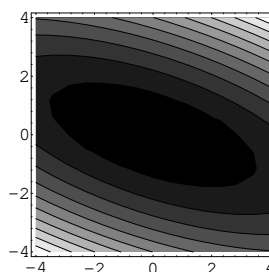
This will be a *sum* of two squares—and hence have a minimum at the origin—when $4 - p^2/4$ is positive. The statement in the exercise therefore should say: The function $f(x, y; p) = x^2 + pxy + 4y^2 \dots$ has a minimum at the origin when $p^2 < 16$ and a minimax when $p^2 > 16$.

17. a. The graph is a saddle.



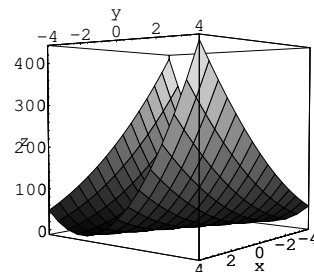
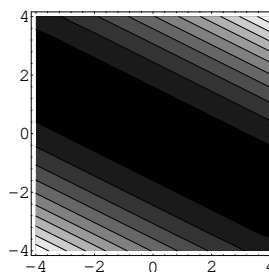
17. b. The contours are hyperbolas, and they lie at fixed levels on the graph. (The hyperbolas are not drawn here, but they are in the solution to exercise 25.) In regions where the hyperbolas are closely-spaced, the graph is steep; in regions where the hyperbolas are widely-spaced, the graph descends slowly.

18. a. The graph has the shape of an elliptical bowl whose sides vary in steepness depending on the direction.



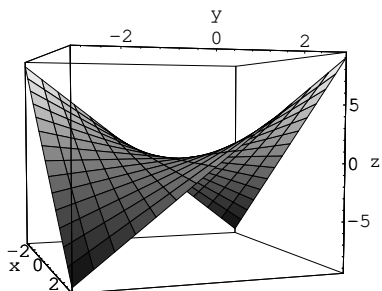
18. b. The contours are ellipses, and they lie at fixed levels on the graph. In regions where the ellipses are closely-spaced, the graph is steep; in regions where the ellipses are widely-spaced, the graph climbs slowly.

19. a. The graph is a trough, or gutter, similar to that in Exercise 15.



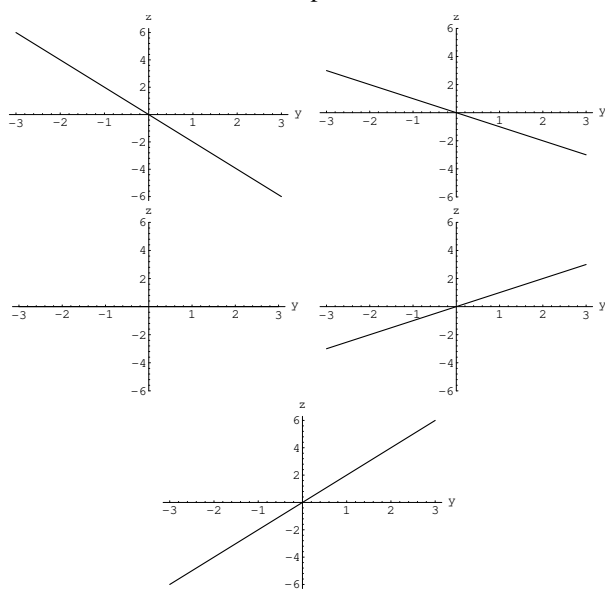
19. b. The contours are pairs of parallel straight lines, and they lie at fixed levels on the graph. As the lines become more closely-spaced, the graph becomes steeper.

20.



Saddle
at $(0,0)$

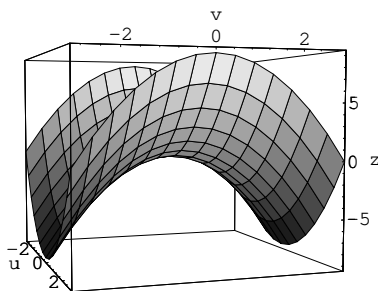
21. a. The slices are straight lines through the origin. When $x = c$, the slice has slope c .



21. b. The results are the same: the slice $y = c$ is a straight line through the origin with slope c . In the previous figures, just relabel the y -axis as the x -axis.

22. The slices in exercise 21 actually appear as the straight lines shown on the graph in exercise 20. It is rather remarkable that a curved surface can have a pair of families of straight lines embedded in it—as we see here.

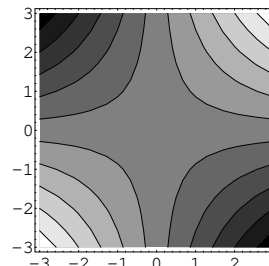
23. a. If $x = u + v$ and $y = u - v$, then $z = xy = u^2 - v^2$.



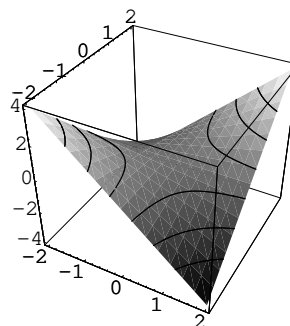
23. b. This graph is also a saddle. However, at first glance it doesn't look too similar to the graph of $z = xy$ in exercise 20. In particular, the "slices" are quite different: they are straight lines in the one but parabolas in the other. Nevertheless, this graph is just the first graph rotated 45° around the z -axis.

24. Yes, a network of straight lines can be drawn on $z = u^2 - v^2$. According to the previous exercise, vertical slices that are made 45° to the u - and v -axes will be straight lines in the surface.

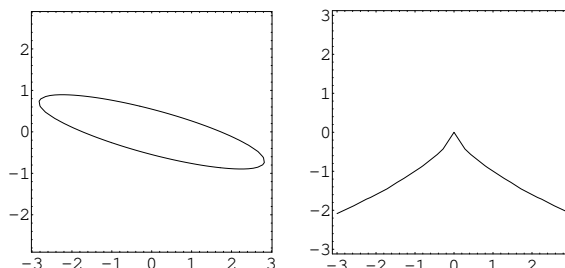
25.



The contours are hyperbolas that lie at fixed levels on the graph. This is illustrated in the figure below. (Notice that the shading is with respect to the original grid; as a consequence, it does not follow the contours exactly.)



26. The curve is the level curve $3 = x^2 + 5xy + 10y^2$ of the function $z = x^2 + 5xy + 10y^2$, so it can be found by plotting level curves. The curve is an ellipse (shown at the left, below).



27. We must plot the level curve $0 = x^2 + y^3$, which can be expressed as $y = x^{2/3}$. The curve has a *cusp* shape, and is shown at the right, above.

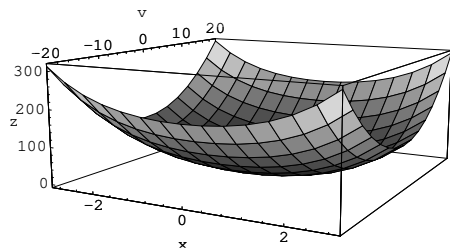
28. The graphs are planes, so their intersection is a straight line. That answers the question. (To identify the line more fully, we can solve the equation

$$2x - y = z = .3x + .8y + 2.3$$

for y in terms of x . The result is $y = 17/18x - 23/18$, a line in the x, y -plane. The line in x, y, z -space lies above this on $z = 2x - y$ (or $z = .3x + .8y + 2.3$.)

First Integrals

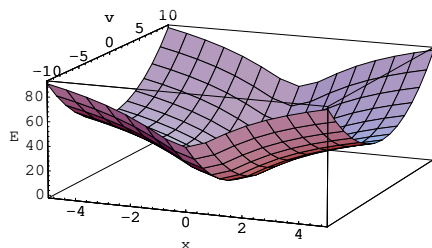
29. a.



The function is $E(x, v) = 8x^2 + \frac{1}{2}x^4 + \frac{1}{2}v^2$. It has a minimum at the origin, but no saddles or maxima. (Note: the v -range of the domain of this function was made much larger than the x -range in order to make the graph look suitably bowl-like.)

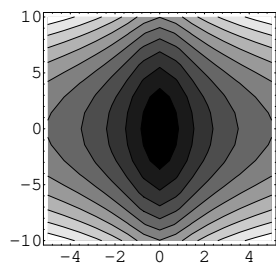
29. b. Minimum energy occurs at the bottom of the graph. The state there is $x = v = 0$; the spring is at rest.

30. a.



The domain chosen to get a good representation of the graph is $-10 \leq x \leq 10$, $-5 \leq v \leq 5$.

30. b.

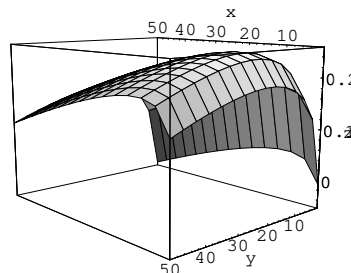


The contour plot shows a minimum at the origin, agreeing with the graph. The fact that the higher contours are not convex ovals (but have indentations) reflects the way the graph opens up as E increases.

30. c. The spring has a state of minimum energy at $(x, v) = (0, 0)$, when the spring is at rest.

30. d. The spring does not have a state of *maximum* energy, because the graph increases upwards without bound. In particular, E can be made as large as we wish by choosing v sufficiently large.

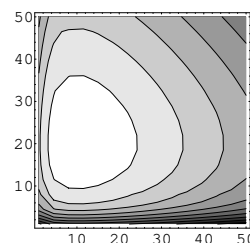
31. a.



We can't enlarge the domain to include zero because E involves $\ln x$ and $\ln y$, and these are undefined when x or y is zero.

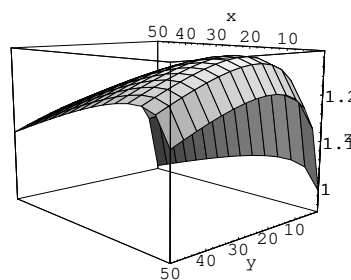
31. b. There is a maximum at $(x, y) \approx (10, 20)$. The surface has no minimum or saddle points. There is no minimum because $E \rightarrow -\infty$ as $x \rightarrow 0$ or $y \rightarrow 0$.

31. c.



The trajectories of the dynamical system are exactly the contours of $E(x, y)$. In other words, the value of E is "conserved" on any trajectory—or any solution of the Lotka-Volterra equations.

32. a.

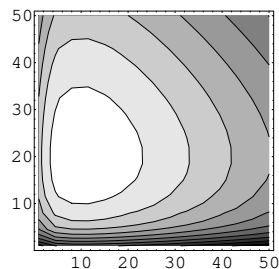


The essential difference between the last two graphs is the vertical (" z ") scale. We have

$$\begin{aligned} \exp(E(x, y)) &= e^{-11\ln y} e^{.04\ln x} \exp(-.005y - .004x) \\ &= \frac{y^{-11} x^{.04}}{\exp(.005y + .004x)} \\ &= H(x, y) \end{aligned}$$

In other words, the value of H at any point (x, y) is just the exponential of the value of E at the same point. The graphs will therefore be very similar; in particular, and extreme of one will be an extreme of the other. You can graph this on $0 \leq x \leq 50, 0 \leq y \leq 50$.

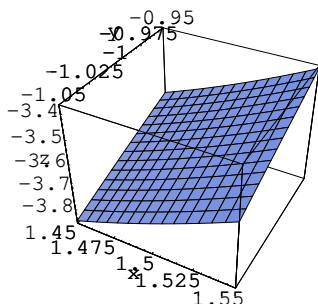
32. b.



The contours of $H(x, y)$ are exactly the same curves as the contours of $E(x, y)$. The only difference is the *values* of the two functions on a given contour.

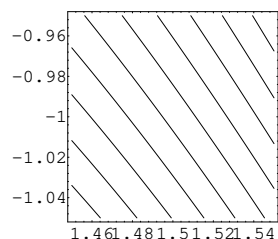
9.2 Local Linearity

1. a.



1. b. By eye, $f(1.45, -1.05) \approx -3.9$, $f(1.55, -1.05) \approx -3.6$, so $\Delta z/\Delta x \approx 3$. Also, $f(1.45, -0.95) \approx -3.7$, so $\Delta z/\Delta y \approx 2$.

2.



The figure above is magnified twice as much as the most magnified figure on page 533—and looks essentially like that one.

3. a. Along the horizontal line $y = -1$ we meet about six bands in the contour plot, so the horizontal spacing between contours is about $.1/6 \approx .017 = \Delta x$.

3. b. In the lower left corner, $z = -3.85$; in the upper right, $z = -3.38$. This is a change of .47, and occurs over 10 contour lines. Therefore, the Δz between adjacent contour lines is $.47/10 = .047$.

3. c. If we take $\Delta x = .017$, then $\Delta z/\Delta x = .047/.017 = 2.76$. (This is remarkably close to the exact value of $\partial z/\partial x = 2.75$.)

4. Along the vertical line $x = 1.5$ we meet about four and a half bands, so the vertical spacing between contours is about $.1/4.5 \approx .022 = \Delta y$. Since we still have $\Delta z = .047$, we find $\Delta z/\Delta y = .047/.022 = 2.14$. (The exact value of $\partial z/\partial y = 2$.)

5. a. The z -intercept is found by setting $x = y = 0$: $z - 3 = -8 - 3 = -11$, so $z = -8$.

5. b. $z = 2x - 3y - 8$.

6. a. The initial-value formula is

$$z - L(1, 4) = \frac{\Delta z}{\Delta x}(x - x_0) + \frac{\Delta z}{\Delta y}(y - y_0)$$

$$\text{so } z = 3(x - 1) - 2(y - 4).$$

6. b. The z -intercept is $z = -3 + 8 = 5$, so the z -intercept form of the original equation is $z = 3x - 2y + 5$.

7. a. $\Delta z = -7\Delta x + 12\Delta y$. Therefore, when (x, y) changes from $(35, 24)$ to $(33, 33)$, $\Delta x = -2$, $\Delta y = 9$ and $\Delta z = 14 + 108 = 122$.

7. b. We are given that $z = 29$ when $(x, y) = (35, 24)$, and we have just shown that $\Delta z = 122$ when (x, y) changes to $(33, 33)$. Therefore, $z = 29 + 122 = 151$ when $(x, y) = (33, 33)$.

When (x, y) changes from $(35, 24)$ to $(0, 0)$, $\Delta x = -35$ and $\Delta y = -24$. Therefore

$$\Delta z = (-7)(-35) + (12)(-24) = -43,$$

so $z(0, 0) = 29 + (-43) = -14$.

7. c. Since $\Delta z/\Delta x = -7$, $\Delta z/\Delta y = +12$, and the z -intercept is -14 , we conclude $z = -7x + 12y - 14$.

8. a. The first, third, and fifth columns of the table contain complete information, and we can use that information to determine a and b in $\Delta z = a\Delta x + b\Delta y$. In fact,

$$10 = -5 \cdot a - 5 \cdot b$$

$$0 = -1 \cdot a - 2 \cdot b,$$

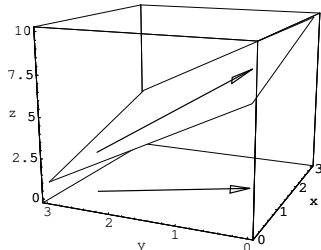
and this implies $b = 2$, $a = -4$. We can now fill in the table.

x	5	7	0	23/4	4	0	4	-18/4
y	9	1	4	6	7	0	16	-1
z	2	-22	12	-7	2	4	20	20

8. b. According to the sixth column, the z -intercept is 4, so the intercept form of the equation is

$$z = -4x + 2y + 4.$$

9. a.

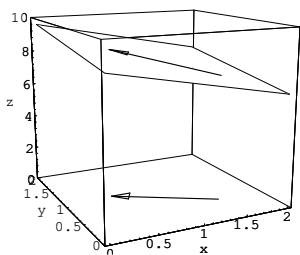


9. b. The slope in the x -direction is 1; it is the coefficient of x in the equation for z . In the graph, it can be seen as the slope of the right edge of the graph in the figure above.

9. c. The slope in the y -direction is -2 , and is shown by the slope of the front edge of the graph above.

10. Yes, the gradient vector $\nabla z = (1, -2)$ points directly “uphill”. The vector on the surface of the plane is $(1, -2, 5)$. The z component is found by computing $\Delta z = 1 \cdot (1) + 2 \cdot (2) = 5$. Both vectors are shown in the figure for exercise 9.

11.



The slope in the x -direction is -1 ; it is seen as the slope of the front edge of the graph above. The slope in the y -direction is $.6$; it is seen as the slope of the side of the graph. The z -intercept of 8.4 ; it is where the front corner of the graph meets the front edge of the cube.

12. Yes, the gradient vector points directly “uphill”. Both vectors are shown in the figure for the previous exercise.

13. We can get the equation from

$$z - 0 = -4(x - 2) + 5(y + 9)$$

$$\text{or } z = -4x + 5y + 53.$$

14. a. If we set $p = \Delta z / \Delta x$ and $q = \Delta z / \Delta y$, then $\Delta z = p\Delta x + q\Delta y$. From $(1, 1, 2)$ to $(0, 5, 4)$ we have

$$\Delta x = -1, \quad \Delta y = 4, \quad \Delta z = 2,$$

while from $(0, 5, 4)$ to $(-3, 0, 12)$ we have

$$\Delta x = -3, \quad \Delta y = -5, \quad \Delta z = 8,$$

This gives two equations in the two unknowns p and q :

$$2 = -1p + 4q$$

$$8 = -3p - 5q.$$

Solving these gives $p = -\frac{42}{17}$ and $q = -\frac{2}{17}$.

14. b. We now know $z = L(x, y) = -\frac{42}{17}x - \frac{2}{17}y + c$, and c is the (as yet unknown) z -intercept. To find c , we can simply substitute into L known values of x , y , and z at a point. If we use $(1, 1, 2)$, we find $2 = -\frac{42}{17} - \frac{2}{17} + c$, so $c = \frac{78}{17}$ is the z -intercept.

14. c. If $(4, 1, c)$ is a point on the graph, then $c = L(4, 1)$, or

$$c = -\frac{42}{17} \cdot 4 - \frac{2}{17} \cdot 1 + \frac{78}{17} = -\frac{92}{17}.$$

14. d. If $(2, 2, 4)$ were a point on the graph, then $L(2, 2) = 4$. But $L(2, 2) = -\frac{10}{17} \neq 4$, so the point is *not* on the graph. Another approach is to compare $(2, 2, 4)$ to the known point $(1, 1, 2)$. We have $\Delta x = \Delta y = 1$, so we must have $\Delta z = -\frac{42}{17} - \frac{2}{17} = -\frac{44}{17}$. But $\Delta z = 2$ in this case, not $-\frac{44}{17}$.

15. a. $L_1(1, 0) = 15$, $L_1(2, 0) = 18$, $L_1(3, 0) = 21$
 $L_1(0, 3) = 24$, $L_1(2, 3) = 30$, $L_1(0, 0) = 12$

15. b. $\Delta L_1 / \Delta x = 3$, $\Delta L_1 / \Delta y = 4$.

16. a. From the basic formula

$$L_1(x, y) = \frac{\Delta L_1}{\Delta x}x + \frac{\Delta L_1}{\Delta y}y + L_1(0, 0)$$

we see $L_1(x, y) = 3x + 4y + 12$. This gives us
 $L_1(3, 2) = 29$, $L_1(7, 0) = 33$, $L_1(7, 7) = 61$,
 $L_1(1.4, 2.9) = 27.8$, $L_1(-2, 9) = 42$,
 $L_1(-10, -100) = -418$.

16. b. $L_1(x, 0) = 3x + 12 = 0$ if $x = -4$ and $L_1(0, y) = 4y + 12 = 0$ if $y = -3$.

17. $L_1(x, y) = 3x + 4y + 12$.

18. a. Along the vertical line $x = 1$ we see $L_2(1, 0) = 20$ and $L_2(1, 4) = 0$. Thus $\Delta L_2 / \Delta y = -20/4 = -5$. Along the bottom ($y = 0$) we see $L_2(1, 0) = 20$ and $L_2(3.5, 0) = 40$. Thus $\Delta L_2 / \Delta x = 20/2.5 = 8$.

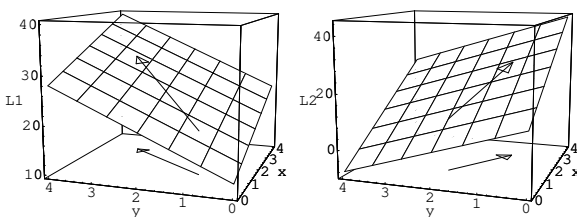
18. b. We now have $L_2(x, y) = 8x - 5y + c$, where c is the z -intercept. We can find this using a known value of L_2 , for example $L_2(1, 0) = 20$. This implies $c = 12$, so $L_2(x, y) = 8x - 5y + 12$.

19. In the figure on the left in the text, the slope in the y -direction is negative, while in the figure on the right it

is positive. Since $\Delta L_2/\Delta y$ is negative while $\Delta L_1/\Delta y$ is positive, the figure on the left is the graph of L_2 .

20. a. $\nabla L_1 = (3, 4); \quad \nabla L_2 = (8, -5).$

20. b.



In the figures above, the gradient vectors have been rescaled (shortened) so they fit. Their directions are unaltered, however, and they point directly uphill in both cases.

21. $\Delta p = (\Delta p/\Delta q) \cdot \Delta q + (\Delta p/\Delta r) \cdot \Delta r$, and the gradient tells us the coefficients: $\Delta p = 5\Delta q - 12\Delta r$. Since we are given $L(9, 15)$ and want to find $L(11, 11)$, $\Delta q = 2$ and $\Delta r = -4$. Therefore $\Delta p = 5 \cdot 2 - 12 \cdot -4 = 58$. Since $L(9, 15) = 17$, $L(11, 11) = 17 + \Delta p = 17 + 58 = 75$.

22. a. $\nabla w = (2, 5).$

22. b. The circle $u^2 + v^2 = 1$ consists of all points that are 1 unit from the origin. According to the discussion in the text (pages 539–541), the largest value of $w(u, v)$ on this set will occur in the direction of the gradient ∇w . We need the unit vector in this direction. It is $(2/\sqrt{29}, 5/\sqrt{29})$. The value of w there is $2(2/\sqrt{29}) + 5(5/\sqrt{29}) = \sqrt{29} \approx 5.385$.

23. a. $z = L(x, y) = -3x + 4y + c$, where c can be chosen arbitrarily—but the student should choose some specific value.

23. b. $\Delta L = -3\Delta x + 4\Delta y = -3 \cdot 2 + 4 \cdot 1 = -2$.

24. a. $L(0, 0) = c$ (where c is the value the student chose in exercise 23).

24. b. The maximum value on the unit circle equals the value at the center of the circle (just determined in part a) plus the length of the gradient vector. The length of ∇z is $\sqrt{9+16} = 5$, so the maximum is $c + 5$, obtained at $(-\frac{3}{5}, \frac{4}{5})$. This value depends on the student's choice of c .

24. c. The minimum value on the unit circle equals the value at the center of the circle (just determined in part a) minus the length of the gradient vector. This is $c - 5$. The difference between the maximum and the minimum is therefore $c + 5 - (c - 5) = 10$, or twice the length of the gradient. This result is independent of the student's choice of c .

25. a. $z(5, 2) = 72$.

25. b. $\Delta z = 7\Delta x + 3\Delta y$. Therefore, if x increases by 2, z increases by 14 (and y doesn't change). To insure that $\Delta z = 0$, we must have $3\Delta y = -14$, so y must change by $-14/3$.

25. c. If $\Delta x = \alpha$ and we want $\Delta z = 0$ then $0 = 7\alpha + 3\Delta y$, so $\Delta y = -7\alpha/3$. This is the trade-off in y .

25. d. If $\Delta y = \beta$, then $0 = 7\Delta x + 3\beta$, so $\Delta x = -3\beta/7$. This is the trade-off in x .

26. a. We have $\Delta L = 5\Delta x - 2\Delta y$, and we are given that $\Delta x = +50$. To find the trade-off in y , we solve $0 = 5 \cdot 50 - 2\Delta y$. Thus $\Delta y = 125$.

26. b. Here $\Delta y = +1$ and $0 = 5\Delta x - 2$, so $\Delta x = 2/5$.

27. a. With linear functions, trade-offs are always in proportion. Since $\Delta x = +1$ implies $\Delta y = -4$, when $\Delta x = -3$ we must have $\Delta y = +12$.

27. b. Since $\Delta y = -4$ implies $\Delta x = +1$, $\Delta y = +10$ implies $\Delta x = +1 \cdot (10/-4) = -5/2$.

28. a. We are given $L(3, 7)$ and asked to find $L(7, 7)$. The difference between these two points is $\Delta x = 4$. Since $\Delta z/\Delta x = 2$, we see $\Delta z = 8$. Since $L(3, 7) = -2$, $L(7, 7) = -2 + 8 = 6$.

28. b. The point here is that $L(3, 7)$ and $L(7, \beta)$ have the same value, so there is a trade-off between x and y . However, we know the trade-off: when x increases by 10, y decreases by 4. Since x increases here by 4, y must decrease $4 \times 4/10 = 1.6$. Therefore $\beta = 7 - 1.6 = 5.4$.

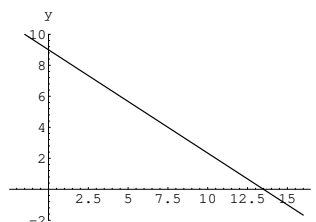
28. c. We now know $L(7, 5.4) = -2$ and $L(7, 7) = 6$. Therefore $\Delta z/\Delta y = (6 - (-2))/(7 - 5.4) = 8/1.6 = 5$.

28. d. At this point we know $L(x, y) = 2x + 5y + c$. Therefore, $-2 = L(3, 7) = 2 \cdot 3 + 5 \cdot 7 + c$, so $c = -43$.

29. We have $\Delta z = -1.5\Delta x - 2.4\Delta y$. If $\Delta x = \alpha$ in a trade-off, then $0 = -1.5\alpha - 2.4\Delta y$, so $\Delta y = -1.5\alpha/2.4 = -5\alpha/8$.

29. a. If both partial slopes change sign, there is no change in the trade-off. The basic equation reads $\Delta z = +1.5\Delta x + 2.4\Delta y$, and when we set $\Delta z = 0$ we have the same equation we had in part a.

30. a.



30. b. Since $z = 2x + 3y + 7 = 34$, we just have the straight line $y = -\frac{2}{3}x + 9$. therefore, if $x = 10$, $y = 7/3$.

30. c. If $\Delta x = 4$, then $\Delta y = -\frac{2}{3} \cdot 4 = -\frac{8}{3}$.

31. a. $w = 3x - 7y - 2z + 1$.

31. b. $\nabla w = (3, -7, -2)$

32. Between $L(3, 0, 5)$ and $L(1, 2, 3)$, $\Delta x = -2$, $\Delta y = 2$, $\Delta z = -2$. Furthermore, $\Delta w = \Delta x - \Delta y + 4\Delta z = -2 - 2 - 8 = -12$. Since $L(3, 0, 5) = 10$, it follows that $L(1, 2, 3) = 10 + \Delta w = 10 - 12 = -2$.

33. Level sets of $w = x + y + z$ are given by the equation $x + y + z = c$, where c is arbitrary. This is the same as the equation $z = -x - y + c$. So the level sets of $w = x + y + z$ are the graphs of $z = -x - y + c$. These graphs are planes in (x, y, z) -space that have slope -1 in the x - and y -directions and an arbitrary z -intercept c . The level sets are therefore parallel planes.

The Microscope Equation

34. The microscope equation for $z = f(x, y)$ at $(x, y) = (a, b)$ is

$$\Delta z \approx \frac{\partial z}{\partial x}(a, b) \cdot \Delta x + \frac{\partial z}{\partial y}(a, b) \cdot \Delta y.$$

Here $\partial f/\partial x = 6x = 12$ and $\partial f/\partial y = 8y = -8$, so the microscope equation is $\Delta z \approx 12\Delta x - 8\Delta y$.

35. a. We start from $f(2, -1) = 3 \times 4 + 4 \times 1 = 16$.

For $f(1.93, -1.05)$ we have $\Delta x = -.07$ and $\Delta y = -.05$. Therefore $\Delta z \approx -.07 \times 12 + .05 \times 8 = -.44$. Hence $f(1.93, -1.05) \approx f(2, -1) + \Delta z = 16 - .44 = 15.56$.

For $f(2.07, -.99)$ we have $\Delta x = .07$ and $\Delta y = .01$. Therefore $\Delta z \approx .07 \times 12 - .01 \times 8 = .76$. Hence $f(2.07, -.99) \approx f(2, -1) + \Delta z = 16 + .76 = 16.76$.

35. b. $f(1.93, -1.05) = 15.5847$ exactly, so the error in the estimate in part a is .0247. $f(2.07, -.99) = 16.7751$ exactly, so the error in the estimate is .0151. The first three digits are accurate in each estimate.

36. At the point $(x, y, z) = (1, 1, \pi)$, $\partial f/\partial x = 2xy \sin z = 0$, $\partial f/\partial y = x^2 \sin z = 0$ and $\partial f/\partial z = x^2 y \cos z = -1$. Therefore the microscope equation is $\Delta f \approx -\Delta z$.

37. The following estimates start from $f(87, 453) = 1254$, and they are based on this microscope equation: $\Delta z = -3.4\Delta x + 4.2\Delta y$.

point	Δx	Δy	Δz	f
(90, 453)	3	0	-10.2	1243.8
(87, 450)	0	-3	-12.6	1241.4
(90, 450)	3	-3	-22.8	1231.2
(100, 500)	13	47	153.2	1407.2

38. a. If $f(87, y) = 1250$, then $\Delta x = 0$ and $\Delta z = -4$. By the microscope equation, $-4 \approx 4.2\Delta y$, so $\Delta y = -4/4.2 = -.95$ and $y = 453 + \Delta y = 452.05$.

38. b. If $f(x, 450) = 1275$, then $\Delta y = -3$ and $\Delta z = 21$. By the microscope equation, $21 \approx -3.4\Delta x - 3 \times 4.2$, so $\Delta x = -(21 + 12.6)/3.4 = -9.88$ and $x = 87 - 9.88 = 77.12$.

39. When $\Delta x = 1$, the trade-off equation becomes $0 = \Delta z - 3.4 \times 1 + 4.2\Delta y$. Therefore, $\Delta y = 3.4/4.2 = .81$.

40. a. At $(27.6, 31.9)$ we make this estimate:

$$\begin{aligned} \frac{\partial Q}{\partial x} &\approx \frac{Q(27.9, 31.9) - Q(27.3, 31.9)}{27.9 - 27.3} \\ &= \frac{15.2 - 15.7}{.6} = -\frac{.5}{.6} = -.833 \end{aligned}$$

40. b. If we use $Q(27.3, 31.9) = 15.7$ to estimate $Q(27, 31.9)$, then $\Delta x = -.3$ and $\Delta y = 0$. Therefore the microscope equation says $\Delta Q \approx -.833\Delta x = -.833 \times -.3 = .25$. Therefore $Q(27, 31.9) \approx Q(27.3, 31.9) + \Delta Q = 15.70 + .25 = 15.95$.

41. First estimate $\partial S/\partial x$ and $\partial S/\partial y$:

$$\begin{aligned} \frac{\partial S}{\partial x} &\approx \frac{S(110, 93) - S(105, 93)}{110 - 105} = \frac{.7}{5} = .14 \\ \frac{\partial S}{\partial y} &\approx \frac{S(105, 95) - S(105, 93)}{95 - 93} = \frac{-.7}{2} = -.35 \end{aligned}$$

Then estimate $S(100, 100)$ from $S(105, 93)$ by the microscope equation. We have $\Delta x = -5$, $\Delta y = 7$, and

$$\Delta S \approx .14\Delta x - .35\Delta y = .14 \times -5 - .35 \times 7 = -3.15.$$

Therefore $S(100, 100) \approx S(105, 93) + \Delta S = 10 - 3.15 = 6.85$.

42. To estimate $f(175, -30, 550)$ we use the microscope equation at the point P . It has the form

$$\Delta f \approx 7\Delta x - 2\Delta y + 5\Delta z.$$

From P to $(175, -30, 550)$, $\Delta x = 2$, $\Delta y = -1$, and $\Delta z = -3$. Therefore $\Delta f = 7 \times 2 - 2 \times -1 + 5 \times -3 = 1$. Finally, $f(175, -30, 550) \approx f(P) + \Delta f \approx 48 + 1 = 49$.

43. a. The equation of the tangent plane comes from the microscope equation at that point. Here $f_x = y$, $f_y = x$; at $(2, -3)$, $f_x = -3$, $f_y = 2$. Therefore $\Delta z = -3\Delta x + 2\Delta y$. Since the tangent plane contains the point $(x, y, z) = (2, -3, -6)$, its equation is $z + 6 = -3(x - 2) + 2(y + 3)$, or $z = -3x + 2y + 6$.

43. b. The z -intercept of $z = xy$ is $z = 0$, while that of the tangent plane is $z = 6$. The tangent plane has a higher z -intercept.

44. a. The coefficients in the microscope equation are the components of the gradient; thus $\nabla H = (2.53, -1.19)$.

44. b. Since H increases most rapidly in the direction of its gradient, the point one unit away from $(35, 26)$ at which H should be largest is in the direction of ∇H but reduced to one unit in length.

Since the length of ∇H is $\sqrt{(2.53)^2 + (-1.19)^2} = \sqrt{7.817} \approx 2.796$, the vector we want is $(\Delta x, \Delta y) = (2.53/2.796, -1.19/2.796) = (.905, -.425)$. Therefore, the point one unit away from $(35, 26)$ at which H is largest is $(35.905, 25.575)$; H is 2.796 units larger there.

45. Since $V_x = 2xy$ and $V_y = x^2$, at the point $(25, 10)$ we have $V_x = 500$ and $V_y = 625$. The microscope equation at $(25, 10)$ is therefore $\Delta V \approx 500\Delta x + 625\Delta y$.

46. a. If the side of the base is x inches and the height is y inches, then the volume of the box is $V = x^2y$ cubic inches. The microscope equation tells us how errors in x and y are propagated to V . Since we assume $x = 25$ and $y = 10$, the microscope equation is the one obtained in the last exercise: $\Delta V \approx 500\Delta x + 625\Delta y$. Here Δx is the error in x , Δy is the error in y , and ΔV is the resulting error in V .

46. b. This is a continuation of the previous question because the volume function here is the same as V there, and the error propagation equation here is the same as the microscope equation there. Finally, x and y are given the same values here as there.

47. We assume here first that $\Delta x/x = .01$ while $\Delta y = 0$, and then we assume $\Delta x = 0$ while $\Delta y/y = .01$. We want to compare the resulting values of $\Delta V/V$. In fact, the result is independent of the starting values of x and y . We have

$$\frac{\Delta V}{V} \approx \frac{2xy\Delta x + x^2\Delta y}{x^2y} = 2\frac{\Delta x}{x} + \frac{\Delta y}{y}.$$

This implies that a 1% error in x produces a 2% error in V , while a 1% error in y produces only a 1% error in V . The error in x has twice the impact of the error in y .

48. We use the formula for the volume of a cone of radius r and height h : $V(r, h) = \pi r^2 h/3$. Therefore, $V(186, 31) = 357492\pi$ cubic meters.

49. The equation for the propagation of relative errors is

$$\begin{aligned} \frac{\Delta V}{V} &\approx \frac{V_r \Delta r + V_h \Delta h}{V} \\ &= \frac{\frac{2}{3}\pi r h \Delta r + \frac{1}{3}\pi r^2 \Delta h}{\frac{1}{3}\pi r^2 h} = 2\frac{\Delta r}{r} + \frac{\Delta h}{h} \end{aligned}$$

(This is essentially the same as the error propagation equation in the last exercise.) It says that a given percentage

error in h produces the same percentage error in V , while the same percentage error in r produces *twice* the percentage error in V .

49. a. By what was just said, there is a 6% error in the calculation of the volume of the basin.

49. b. By what was just said, there is a 5% error in the calculation of the volume.

49. c. The errors add, so there is an 11% error (or uncertainty) in the calculation of the volume.

50. From the relative error propagation equation for V in the previous solution, we know a 2% error in the radius will lead to a 4% error in the volume. If the volume error is to be no more than 5%, then the error due to the error in height (depth) can contribute no more than 1%. In fact, the error in depth can be no more than 1%.

51. A 3% error in the radius will lead to a 6% error in the volume. This already exceeds the limit of a 5% total error in the volume—even if the depth were to be measured no error whatsoever. Therefore it is impossible to measure the depth accurately enough to have the volume known to within 5%.

52. We need a formula for the volume W of the bowl in terms of r and h . Here is one way to get it. Let z measure distance from the bottom of the bowl and let x measure horizontal distance out from the center. Then the bowl is the graph of a function $z = kx^2$ for some k . Since $z = h$ when $x = r$, we have $h = kr^2$, so $k = h/r^2$ and $z = hx^2/r^2$. This means that the horizontal slice of the bowl at height z is a circle of radius $x = r\sqrt{z/h}$ and area $\pi x^2 = \pi r^2 z/h$. From exercises 13–16 in section 6.2 (Volume as a Riemann sum) we can see that the volume W we want is just the integral of the area slices:

$$W(r, h) = \frac{\pi r^2}{h} \int_0^h z dz = \frac{\pi r^2}{h} \frac{h^2}{2} = \frac{1}{2} \pi r^2 h.$$

The volume of the bowl is $W(186, 31) = 536238\pi$. It is larger than the volume of the cone with the same r and h . In fact, $2W(r, h) = \pi r^2 h = 3V(r, h)$, or $W(r, h) = \frac{3}{2}V(r, h)$.

53. The error propagation equation for W is the same as the one for V :

$$\begin{aligned} \frac{\Delta W}{W} &\approx \frac{W_r \Delta r + W_h \Delta h}{W} \\ &= \frac{\pi r h \Delta r + \frac{1}{2}\pi r^2 \Delta h}{\frac{1}{2}\pi r^2 h} = 2\frac{\Delta r}{r} + \frac{\Delta h}{h} \end{aligned}$$

A 3% error in the radius means $\Delta r/r = 3\%$ and a 5% error in the depth means $\Delta h/h = 5\%$. All this implies that

$\Delta W/W \approx 2 \cdot 3\% + 5\% = 11\%$, so there is an 11% error in the calculated capacity.

54. From the error propagation equation we see a 2% error in r leads to a 4% error in W . Furthermore, as long as the error in h is no more than 1%, the total error in W will not exceed 5%.

55. The value of $E(\pi/2, 2) = 1 + 2 = 3$. Since $E_x = \sin x = 1$ and $E_v = v = 2$, the error propagation equation is $\Delta E \approx 1\Delta x + 2\Delta v$. (We have used the explicit values for x , v , and E here because the general form doesn't simplify the way it did in earlier examples.) We are given that $\Delta x = .05 \cdot \pi/2 \approx .079$ and $\Delta v = .10 \cdot 2 = .2$. Therefore, $\Delta E/E \approx (.079 + 2 \cdot .2)/3 \approx .16$. In other words, the value $E = 3$ is known to within a possible error of 16%.

56. a. In a trade-off, Δx and Δv change in such a way as to make $\Delta E = 0$, where $\Delta E \approx 1\Delta x + 2\Delta v$ (the microscope equation). We assume x decreases, so $\Delta x < 0$ and $\Delta v > 0$ in order to keep $\Delta E = 0$. Therefore v must *increase* to conserve energy.

56. b. If $\Delta E = 0$, then the microscope equation implies $\Delta v = -\frac{1}{2}\Delta x$. We assume that x decreases by $\pi/180$, so $\Delta x = -\pi/180$ and $\Delta v = -\frac{1}{2} \cdot -\pi/180 = +\pi/360$.

Linear Approximations

57. a. We have $f_x = \cos y \cos x$ and $f_y = -\sin x \sin y$; at $(x, y) = (0, \pi/2)$, these become $f_x = 0$ and $f_y = 0$. Since $f(0, \pi/2) = 0$ as well, the equation of the linear approximation is

$$z - f(0, \frac{\pi}{2}) = f_x(0, \frac{\pi}{2})(x - 0) + f_y(0, \frac{\pi}{2})(y - \frac{\pi}{2})$$

or just $z = 0$.

57. b. The equation of the tangent plane is the same as the linear approximation equation: $z = 0$.

57. c. The tangent plane *is* the x, y -plane; the two meet everywhere.

58. $w - 2 = -1(x - 3) + 3(y - 4)$, or $w = -x + 3y - 7$.

59. a. $\phi_x = 6x + 7y - 5$, $\phi_y = 7x - 4y + 3$, so $\phi_x(a, b) = 6a + 7b - 5$, $\phi_y(a, b) = 7a - 4b + 3$. The equation of the tangent plane is therefore

$$\begin{aligned} z - \phi(a, b) &= (6a + 7b - 5)(x - a) + (7a - 4b + 3)(y - b) \\ z &= (6a + 7b - 5)x + (7a - 4b + 3)y - 3a^2 - 7ab + 2b^2 \end{aligned}$$

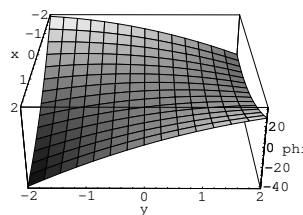
Algebraic manipulation of the first equation (and the fact that $\phi(a, b) = 3a^2 + 7ab - 2b^2 - 5a + 3b$) yields the second equation.

59. b. The tangent plane will be horizontal when $\phi_x = \phi_y = 0$. Thus we must solve the simultaneous equations

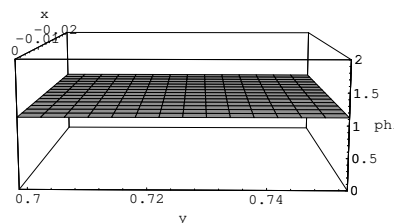
$$\begin{aligned} 6a + 7b - 5 &= 0 \\ 7a - 4b + 3 &= 0 \end{aligned}$$

There is one solution: $(a, b) = (-1/73, 53/73)$.

59. c. Here is the graph of $\phi(x, y)$ over $-2 \leq x \leq 2$, $-2 \leq y \leq 2$:



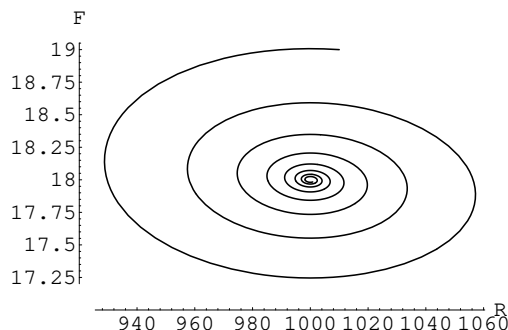
Here is a magnification; we now have $-2/73 \leq x \leq 0$ and $51/73 \leq y \leq 55/73$. This graph appears horizontal as well as flat.



60. To confirm that $(R, F) = (1000, 18)$ is an equilibrium, we need only evaluate the right hand sides of the differential equations and show that they are 0:

$$\begin{aligned} .1(1000) - .00001(1000)^2 - .005(1000)(18) &= 100 - 10 - 90 = 0 \\ .00004(1000)(18) - .04(18) &= .04(18) - .04(18) = 0 \end{aligned}$$

Below is a plot of a trajectory near the equilibrium. It shows the equilibrium is a spiral attractor.



61. We have

$$g_{1R} = .1 - .00002R - .005F = -.01$$

$$g_{1F} = -.005R = -5$$

$$g_{2R} = .00004F = .00072$$

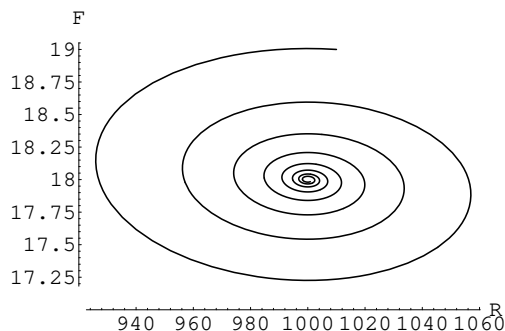
$$g_{2F} = .00004R - .04 = 0$$

Since $g_1(1000, 18) = g_2(1000, 18) = 0$, the linear approximations become

$$\ell_1(R, F) = -.01(R - 1000) - 5(F - 18)$$

$$\ell_2(R, F) = .00072(R - 1000)$$

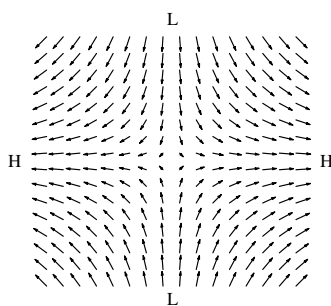
62.



It is clear from the formulas that $\ell_1(1000, 18) = \ell_2(1000, 18) = 0$, so the new system does have an equilibrium at $(1000, 18)$. Furthermore, the portrait is virtually the same as the original portrait of the non-linear (Lotka-Volterra) system at the same point. Both are spiral attractors.

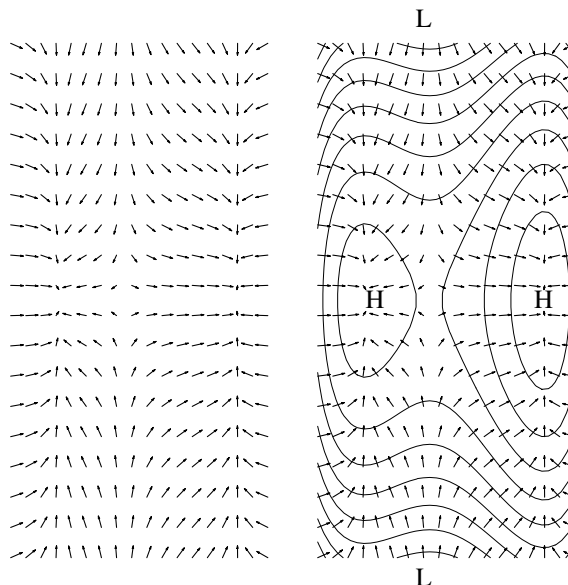
The Gradient Field

63. a.



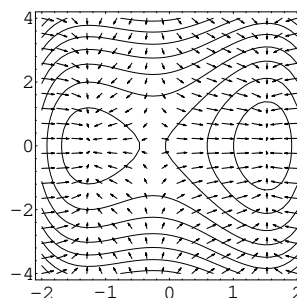
63. b. The maximum values are found near H (along the x-axis) and the minimum values near L (along the y-axis).

64. The gradient field is $\nabla f = (2 + 8x - 4x^3, -2y)$. It is shown by itself on the left on the next page. The location of maximum and minimum values is indicated by H and L on the accompanying figure which shows contours for f as well.



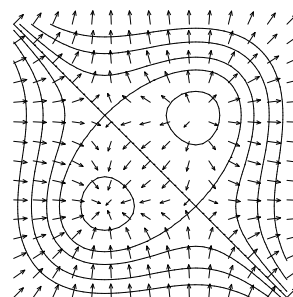
65. The contour lines and gradient vector field of $f(x, y) = 2x + 4x^2 - x^4 - y^2$ are shown at the right, above. Each vector is indeed perpendicular to the contour line which passes through its base. The figures are so elongated because horizontal and vertical scales are kept equal; in the statement of the problem, the y-range is specified to be twice the size of the x-range.

Mathematica warning: if, in a Show command, the contour plot is given first and the vector field second, the following figure will be produced:



In this diagram the gradient vectors are no longer perpendicular to the contours because the scales along the axes are now unequal. Thus lines with negative reciprocal slopes will no longer plot as perpendicular lines.

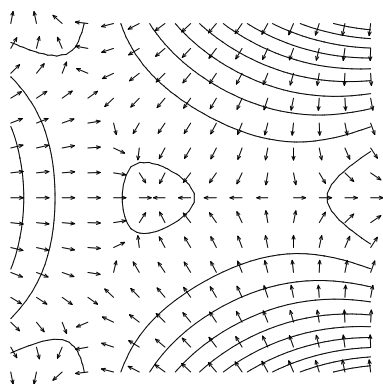
66.



The vector field is actually the gradient field of the function $x^3 + y^3 - 3x - 3y$. The field and the contours are produced by the following Mathematica code:

```
<<Graphics`PlotField`
ex1 = PlotGradientField[
  x^3 + y^3 - 3 x - 3 y,
  {x, -3, 3}, {y, -3, 3},
  ScaleFunction -> (#/(#+1)&),
  ScaleFactor -> .3
];
pl1 = ContourPlot[
  x^3 + y^3 - 3 x - 3 y,
  {x, -3, 3}, {y, -3, 3},
  ContourShading -> False,
  Contours -> {-14, -8, -3, 0, 3, 8, 14},
  PlotPoints -> 40,
  Frame -> None
];
Show[ex1, pl1];
```

67.

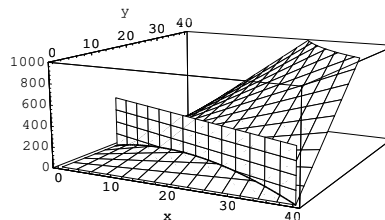


The contours are actually those of the function $g(x, y) = x^3 - 3xy^2 - 10x^2 - 10y^2$. The contours and the gradient field are produced by the following Mathematica code:

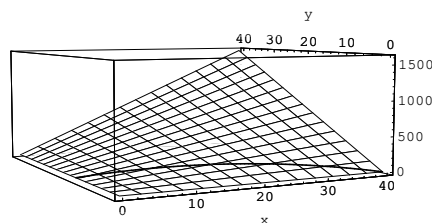
```
<<Graphics`PlotField`
pl2 = ContourPlot[
  x^3 - 3 x y^2 - 10 x^2 - 10 y^2,
  {x, -8, 12}, {y, -10, 10},
  ContourShading -> False,
  PlotPoints -> 40,
  Frame -> None
];
ex2 =
PlotGradientField[
  x^3 - 3 x y^2 - 10 x^2 - 10 y^2,
  {x, -8, 12}, {y, -10, 10},
  ScaleFunction -> (#/(#+1)&),
  ScaleFactor -> .7
];
Show[ex2, pl2];
```

9.3 Optimization

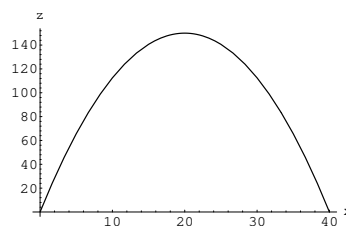
1. The constraints $3x + 8y \leq 120$, $x \geq 0$, $y \geq 0$, define the portion of the first quadrant on or below the line $3x + 8y = 120$.



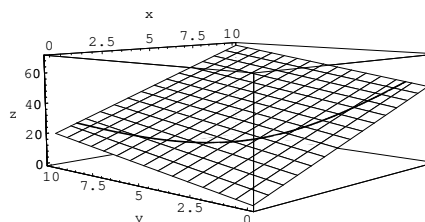
If we erect a vertical “curtain” over this line, it cuts off the graph of $z = xy$ and we see above that the maximum value of z lies along the curtain. Here is a second view; the curve marks where the vertical curtain cuts the graph.



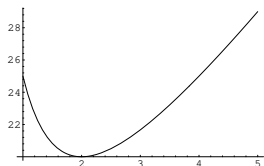
It is evident that there is a maximum somewhere in the interval $15 \leq x \leq 25$. We can determine the curve by solving $3x + 8y = 120$ for y ($y = -\frac{3}{8}x + 15$) and substituting this into $z = xy$. The result is the ordinary parabola $z = x(15 - \frac{3}{8}x) = x(120 - 3x)/8$. It has roots at $x = 0$ and $x = 40$. Its maximum is half-way between them, at $x = 20$. The maximum value is $z = 150$.



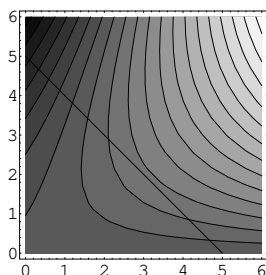
2. The constraints $x \geq 0$, $y \geq 0$, $xy \leq 10$ define the portion of the first quadrant on or outside the hyperbola $xy = 10$. The following figure shows that the minimum value of the linear function $z = 5x + 2y$ lies on this hyperbola.



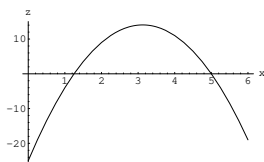
From the figure it is clear that the minimum occurs when x is somewhere in the interval $1 \leq x \leq 5$. To determine x exactly, we solve $xy = 10$ for y ($y = 10/x$) and substitute this into the formula for z : $z = 5x + 20/x$. This graph is plotted below; it shows the minimum occurs when $x = 2$ and the minimum value of z is 20.



3.



The maximum occurs at the lightest point in the triangle bounded by the two axes and the diagonal line $x + y = 5$. As the plot shows, this point is actually on the constraint line $x + y = 5$, very near where $x = 3$. We can determine the point more exactly by substituting $y = 5 - x$ into the formula for z : $z = 3x(5 - x) - (5 - x)^2 = -4x^2 + 25x - 25 = (5 - x)(4x - 5)$. This graph is an ordinary parabola with roots at $x = 5$ and $x = 5/4$.

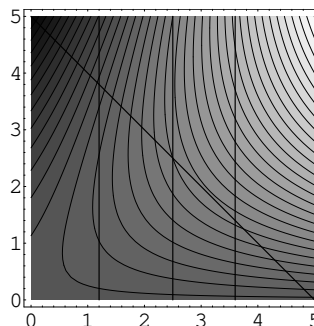


Its maximum occurs half-way between, at $x = 25/8$ (and $y = 15/8$). The maximum value of z is therefore $z = 225/16$.

4. The answer involves *three* cases for the parameter a . The following table gives the location (x, y) of the maximum for the different ranges of values for a :

parameter	(x, y)	location of maximum
$\frac{25}{8} \leq a$	$(25/8, 15/8)$	along diagonal line
$2 \leq a \leq \frac{25}{8}$	$(a, 5 - a)$	at vertex
$0 \leq a \leq 2$	$(a, 3a/2)$	along vertical line

If $\frac{25}{8} \leq a$, then the point $(25/8, 15/8)$ satisfies the constraint $x \leq a$. Therefore this constraint imposes no further limitation and the maximum continues to be at $(25/8, 15/8)$.

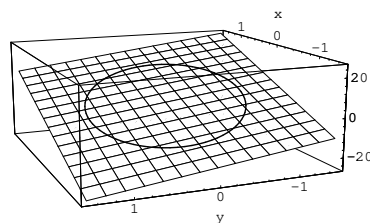


Now suppose $a \leq \frac{25}{8} = 3.125$. Then the point $(25/8, 15/8)$ fails to satisfy the constraint $x \leq a$, so we must look elsewhere for the maximum. In the figure above, the line $x = a$ is drawn for $a = 1.2, 2.5$, and 3.6 . It shows that the maximum (which is always at the point where the shading is lightest) lies *somewhere* along the vertical line when $a \leq \frac{25}{8}$.

Notice that, if a isn't too small, the maximum is at the point where the vertical line meets the diagonal line; but once a drops below a certain value (that appears to be around 2), the maximum leaves the diagonal line and "slides down" the vertical line. In fact, the maximum is now at a point where the vertical line is tangent to a contour.

To see where the maximum leaves the diagonal line, we will determine where the maximum lies on the vertical line. We have $x = a$ so $z = 3ay - y^2 = y(3a - y)$. This is an ordinary parabola with roots at $y = 0$ and $y = 3a$; the maximum is half way between, at $y = 3a/2$. This is the point $(a, 3a/2)$. Since $x + y = 5a/2$ for this point, this point will lie below the line $x + y = 5$ precisely when $a < 2$. This completes the explanation of the table.

5. Points (x, y) that are one unit from the origin satisfy the condition $x^2 + y^2 = 1$, so we are looking for the extremes of the function $z = 12x - 5y$ subject to the constraint $x^2 + y^2 = 1$. From the figure below we see the maximum occurs when x is near 1 and the minimum when x is near -1 .



We can find the extremes more exactly by describing a point on the unit circle using the circular functions: $x = \cos t$, $y = \sin t$. Then z becomes a function of t : $z(t) = 12\cos t - 5\sin t$. At a critical point of this function (where

$z'(t) = 0$ we have

$$-12 \sin t - 5 \cos t = 0 \quad \text{or} \quad \frac{\sin t}{\cos t} = -\frac{5}{12}.$$

Since $12^2 + 5^2 = 13^2$, the points $(\pm 12/13, \pm 5/13)$ are on the unit circle. In particular, we have the following solutions of the critical point equation:

maximum	minimum
$x = \cos t = 12/13$	$x = \cos t = -12/13$
$y = \sin t = -5/13$	$y = \sin t = 5/13$
$z = 13$	$z = -13$

6. We can analyze this problem the same way we analyzed the previous one. We describe a point on the unit circle with $x = \cos t$, $y = \sin t$. Then $z = px + qy$ becomes a function of t : $z(t) = p \cos t + q \sin t$. At a critical point we have

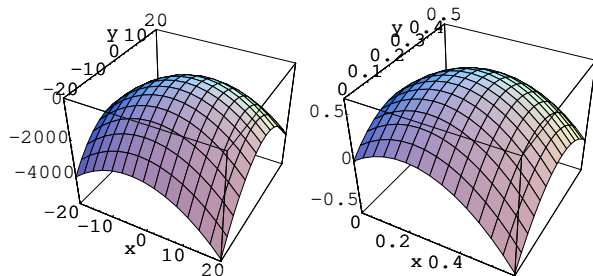
$$z' = -p \sin t + q \cos t = 0 \quad \text{or} \quad \frac{\sin t}{\cos t} = \frac{q}{p}.$$

Let $p^2 + q^2 = r^2$. Then the points $(\pm p/r, \pm q/r)$ are on the unit circle. In particular, we have the following solutions of the critical point equation:

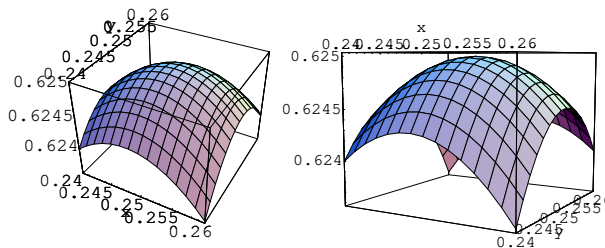
maximum	minimum
$x = \cos t = p/r$	$x = \cos t = -p/r$
$y = \sin t = q/r$	$y = \sin t = -q/r$
$z = r = \sqrt{p^2 + q^2}$	$z = -r$

Notice that this works whether or not p or q is negative.

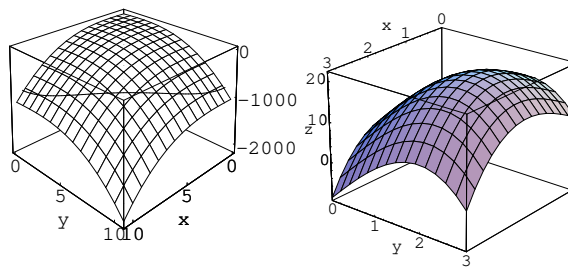
7. The spirit of this question is to inspect the graph of $z = 2xy - 5x^2 - 7y^2 + 2x + 3y$ at higher and higher magnifications. The first view, below, shows a maximum, and the second starts to find it.



The next two figures are two different views at the same magnification level. From the figures it is clear that the maximum is achieved near the point $(x, y) \approx (0.25, 0.25)$ and the maximum value is .625, to three decimal places accuracy.



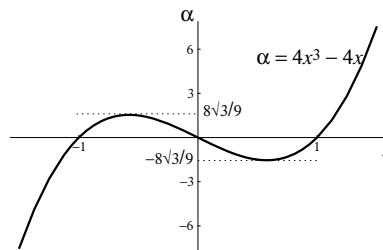
8. The constraint $x^2 + y^2 = 100$ traces out an interesting closed curve on the graph of $z = 6x + 12y - x^3 - y^3$ (left, below). It is clear that the highest point on the graph (the location of the maximum value of z) is at the back. When we magnify the graph (on the right), we see there is a maximum at a critical point. The critical point is $(\sqrt{2}, 2)$, and $z = 16 + 4\sqrt{2} = 21.6569$ there.



9. a. The critical points of $z = x^4 - 2x^2 - \alpha x + y^2$ simultaneously satisfy

$$\frac{\partial z}{\partial x} = 4x^3 - 4x - \alpha = 0 \quad \frac{\partial z}{\partial y} = 2y = 0.$$

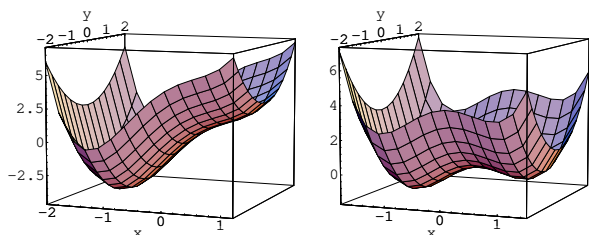
Therefore they lie on the x -axis (because $y = 0$). The equation $\partial z / \partial x = 0$ defines α as a function of x : $\alpha = 4x^3 - 4x$, and this is graphed below.



For a given α , the x -coordinates of the critical points of z are the points where a horizontal line $\alpha = c$ meets this cubic curve. Sometimes there is only one x ; other times there are three.

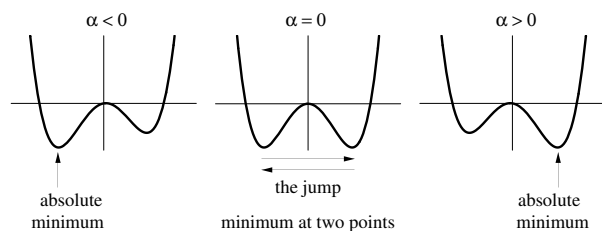
The figure indicates there are three roots when $-8\sqrt{3}/9 < \alpha < 8\sqrt{3}/9$. To see that this is true, note that the extremes of the curve $\alpha = 4x^3 - 4x$ occur where $\alpha' = 12x^2 - 4$, that is, where $x = \pm\sqrt{3}/3$. At these points, $\alpha = 4(\pm\sqrt{3}/3)^3 - 4(\pm\sqrt{3}/3) = \mp 8\sqrt{3}/9$.

When α is outside the range $[-8\sqrt{3}/9, 8\sqrt{3}/9]$, the single root yields a minimum of $z = x^4 - 2x^2 - \alpha x + y^2$. When α is inside, there are two minima and a saddle between them. The following figures illustrate this with $\alpha = -3$ and $\alpha = -0.5$.

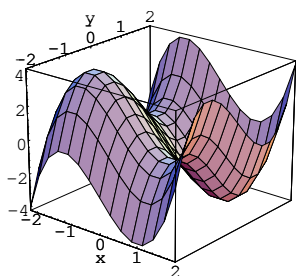


Of course, when there are two (local) minima, only one of them can be the actual (absolute) minimum. When $\alpha = -0.5$, the figure shows that the absolute minimum occurs at a point where $x < 0$. When $\alpha = 0$ the graph of z is symmetric (because, for example, the formula then has only even powers in it) and the two minima are at exactly the same depth. The minima occur at the points where $x = \pm 1$; the minimum value is $z = -2$.

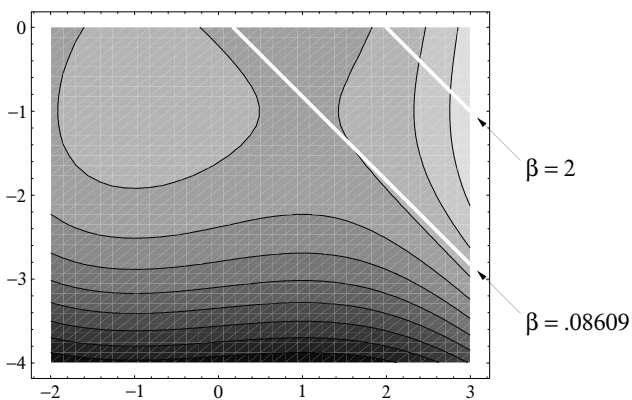
9. b. The catastrophe that the exercise refers to involves a jump in the location of the absolute minimum. This happens as α passes through 0. Since all the critical points lie on $y = 0$, we can see what happens simply by looking at slices of the graph.



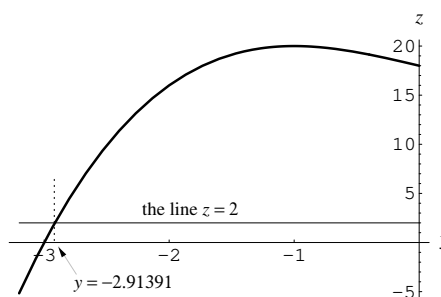
10. a. First find the critical points of $z = x^3 + y^3 - 3x - 3y$. They are determined by the equations $z_x = 3x^2 - 3 = 0$ and $z_y = 3y^2 - 3 = 0$, so they are these four points: $(1, 1)$, $(1, -1)$, $(-1, 1)$, $(-1, -1)$. The following graph shows that the first is a local minimum, the next two are saddles, and the last is a local maximum. The value of z at this maximum is $z = +2$.



The constraints $x \leq 3$, $y \leq 0$, $x + y \leq \beta$ tell us to consider only points in the third and fourth quadrants that lie to the left of $x = 3$ and below the diagonal line $x + y = \beta$. (The diagonal line has no effect until $\beta < 3$.) In the contour plot below, the maximum occurs at the lightest point. The plot makes it clear that that point is somewhere along the right side, on the line $x = 3$.

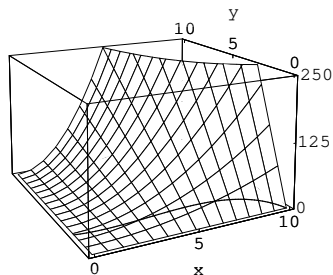


On the line $x = 3$, $z = z(3, y) = y^3 - 3y + 18$. When $y \leq 0$, the maximum is at $y = -1$, and $z = 20$ there. Therefore, so long as the line $x + y = \beta$ lies above the point $(x, y) = (3, -1)$, z will attain its maximum at this point. The line $x + y = \beta$ passes through this point when $\beta = x + y = 3 - 1 = 2$. Therefore, we can say z has its maximum at $(3, -1)$ when $\beta \geq 2$. When $\beta < 2$, the maximum is at the point where the constraint line $x + y = \beta$ meets the constraint line $x = 3$, namely, at $(x, y) = (3, \beta - 3)$. This continues until the value of z drops to 2; then the location of the maximum jumps to $(-1, -1)$.

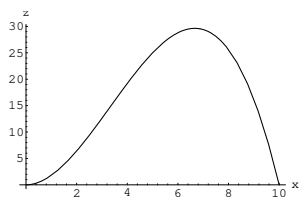


10. b. The jump happens when y satisfies the condition $y^3 - 3y + 18 = 2$, or $y = -2.91391$. Then $\beta = \beta_0 = 3 - 2.91391 = .08609$. The two critical constraint lines are shown on the contour plot, and the graph of z as a function of y on the slice $x = 3$ is shown directly above.

11. Shown below is the graph of $z = x^2y$ and the curve along which the constraint line $x + 5y = 10$ slices it in the first quadrant. It is clear that the maximum occurs when x is somewhere between 5 and 10.



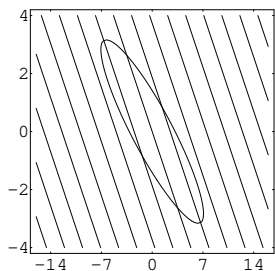
If we write the constraint in the form $y = -\frac{1}{5}x + 2$, then $z = -\frac{1}{5}x^3 + 2x^2$, which we graph as an ordinary curve in the x, z -plane.



We can find the maximum on this curve by setting $z' = 0$, where $z' = -\frac{3}{5}x^2 + 4x$. The solutions are $x = 0$ and $x = 20/3$. Therefore the maximum value of z is $z(20/3, 2/3) = 800/27 \approx 29.6296$.

12. One way to answer this question is with Lagrange multipliers. The constraint curve $g(x, y) = x^2 + 4xy + 5y^2 - 10 = 0$ is an ellipse, and

$$\frac{\partial g}{\partial x} = 2x + 4y \quad \frac{\partial g}{\partial y} = 4x + 10y$$



Since $f(x, y) = 3x + 4y$, $\partial f / \partial x = 3$, $\partial f / \partial y = 4$, and the three Lagrange multiplier equations are

$$\begin{aligned} x^2 + 4xy + 5y^2 - 10 &= 0 \\ 2x + 4y &= 3\lambda \\ 4x + 10y &= 4\lambda \end{aligned}$$

Subtracting twice the second equation from the third, we see that $2y = -2\lambda$, or $y = -\lambda$. Substituting this into the second equation, we find $x = 7\lambda/2$. Finally, substituting these values into the first equation, we see

$$\frac{49}{4}\lambda^2 + 4 \cdot \frac{7}{2} \cdot -1 \cdot \lambda^2 + 5\lambda^2 - 10 = 0,$$

which reduces to $\lambda^2 = 40/13$. Hence $\lambda = \pm 2\sqrt{10/13}$, implying that

$$\text{maximum: } (x, y) = (7\sqrt{10/13}, -2\sqrt{10/13})$$

$$\text{minimum: } (x, y) = (-7\sqrt{10/13}, 2\sqrt{10/13})$$

The contour plot makes it clear that the first is the maximum and the second is the minimum.

Another approach is to parametrize the constraint ellipse and then evaluate $f(x, y)$ with x and y given in terms of the parameter.

By completing the square we can write

$$x^2 + 4xy + 5y^2 = (x + 2y)^2 + y^2 = 10.$$

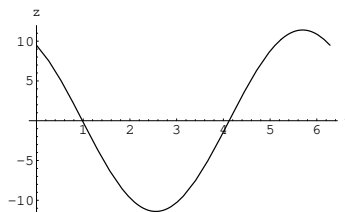
This is a “circle” of radius $\sqrt{10}$ in the variables $x + 2y$ and y . We can parametrize this circle using the sine and cosine functions:

$$x + 2y = \sqrt{10} \cos t \quad y = \sqrt{10} \sin t.$$

Therefore $x = (x + 2y) - 2y = \sqrt{10} \cos t - 2\sqrt{10} \sin t$.

We now substitute these expressions in for x and y in $f(x, y) = 3x + 4y$ and get

$$\begin{aligned} f(x(t), y(t)) &= 3(\sqrt{10} \cos t - 2\sqrt{10} \sin t) + 4(\sqrt{10} \sin t) \\ &= \sqrt{10}(3 \cos t - 2 \sin t). \end{aligned}$$



We can find the extremes of this function (shown above) by differentiating $f(x(t), y(t))$ with respect to t :

$$f(x(t), y(t))' = -\sqrt{10}(3 \sin t + 2 \cos t) = 0$$

when $\cos t = -\frac{3}{2} \sin t$. Therefore, $1 - \sin^2 t = \cos^2 t = \frac{9}{4} \sin^2 t$, so $1 = \frac{13}{4} \sin^2 t$ and $\sin t = \pm 2/\sqrt{13}$.

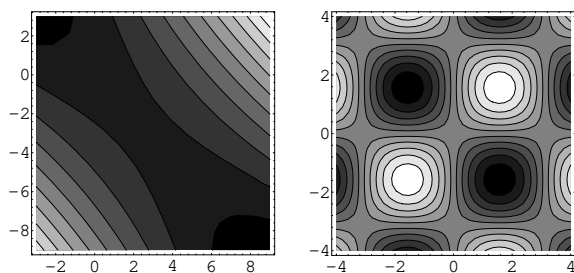
Therefore $y = \pm 2\sqrt{10/13}$; with some further algebra we find $x = \mp 7\sqrt{10/13}$, and we have arrived at the same conclusion we reached earlier using Lagrange multipliers.

13. a. The partial derivative equations are linear

$$z_x = 6x + 7y + 5 = 0$$

$$z_y = 7x + 4y - 6 = 0,$$

and they have only one solution: $x = 62/25$, $y = -72/25$. The contour plot (below left) shows that this critical point is a saddle.



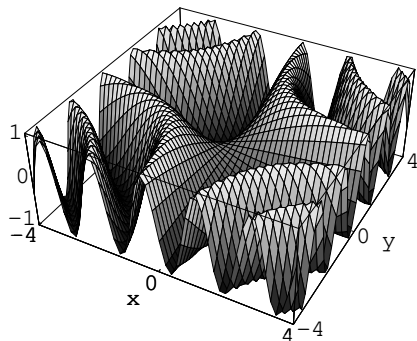
13. b. The contour plot of $z = \sin x \sin y$ (above right) shows two maxima, two minima and nine saddles. Since the partial derivative equations are $z_x = -\cos x \sin y = 0$ and $z_y = -\sin x \cos y = 0$, the critical points are on the lines $x = k\pi/2, y = m\pi/2$, where k and m are integers.

By inspecting the plot we can see that the saddles are at $(-2\pi, -2\pi), (-2\pi, 0), (-2\pi, 2\pi), (0, -2\pi), (0, 0), (0, 2\pi), (2\pi, -2\pi), (2\pi, 0),$ and $(2\pi, 2\pi)$. The maxima are at $(-\pi/2, -\pi/2)$ and $(\pi/2, \pi/2)$. The minima are at $(-\pi/2, \pi/2)$ and $(\pi/2, -\pi/2)$.

13. c. The extremes satisfy $z_x = -y \cos xy = 0, z_y = -x \cos xy = 0$, which implies $y = 0$ or $\cos xy = 0$ and $x = 0$ or $\cos xy = 0$. If $\cos xy = 0$ then $xy = \frac{\pi}{2} + k\pi$ for k an integer. Altogether, this means that $(0, 0)$ is a critical point, while each hyperbola

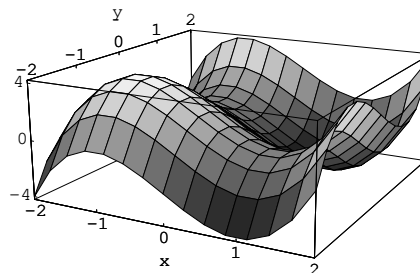
$$y = \frac{(\pi/2 + k\pi)}{x}, \quad k \text{ an integer}$$

consists entirely of critical points. The graph below shows that the origin is a saddle and the critical curves alternate between maxima and minima.

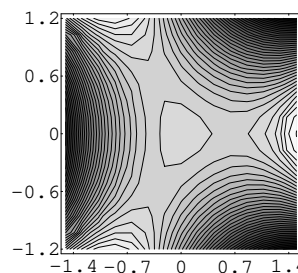


13. d. Since $z_x = 2x \exp(x^2 + y^2)$ and $z_y = 2y \exp(x^2 + y^2)$, and since $\exp u > 0$ for all u , the only point where $z_x = z_y = 0$ is $(x, y) = (0, 0)$. This critical point is a minimum because $\exp(0) = 1$ while $\exp(x^2 + y^2) > 1$ for any $(x, y) \neq (0, 0)$.

13. e. $z_x = 3x^2 - 3 = 0$ if $x = \pm 1$, and $z_y = 3y^2 - 3 = 0$ if $y = \pm 1$. Hence the critical points are $(1, 1)$ (saddle), $(1, -1)$ (minimum), $(-1, 1)$ (maximum), $(-1, -1)$ (saddle). The graph helps us identify the type of each critical point.



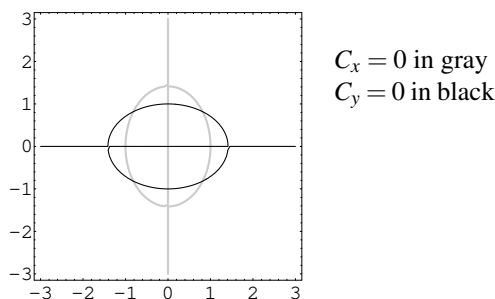
13. f. $z_x = 3x^2 - 3y^2 - 2x = 0$ and $z_y = -6xy - 2y = -2y(3x + 1) = 0$. The second equation implies either $y = 0$ or $x = -1/3$. If we substitute $y = 0$ into the first equation, we find $x(3x - 2) = 0$. This implies $x = 0$ or $x = 2/3$. Therefore $(0, 0)$ and $(2/3, 0)$ are critical points. If, instead, we substitute $x = -1/3$ into the first equation, we find $1 - 3y^2 = 0$. This implies $y = \pm 1/\sqrt{3}$. Therefore $(-1/3, 1/\sqrt{3})$ and $(-1/3, -1/\sqrt{3})$ are critical points—making four altogether. The contour plot below shows that $(0, 0)$ is a local maximum and the other three are saddles. (Note that we need a lot of contours in order to detect the maximum at the origin.)



14. a. $C(x, y) = 1 - 2x^2 + x^4 - 2y^2 + x^2y^2 + y^4$, so

$$C_x = -4x + 2y^2x + 4x^3 = 2x(2x^2 + y^2 - 2) = 0$$

$$C_y = -4y + 2x^2y + 4y^3 = 2y(x^2 + 2y^2 - 2) = 0$$



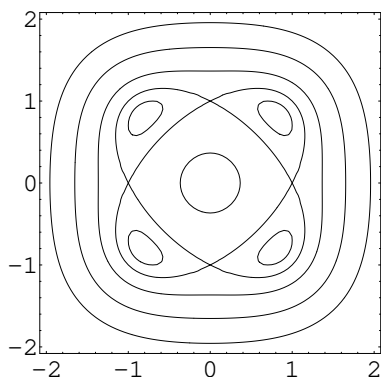
A critical point is one that is both gray and black. There are nine such points: four points where the gray ellipse meets the black; two where the gray line meets the black ellipse; two where the gray ellipse meets the black line; and one where the gray and black lines meet.

By symmetry, the gray and black ellipses meet on the lines $y = \pm x$. Substituting this equation into $2x^2 + y^2 - 2 = 0$ yields $3x^2 = 2$ or $x = \pm\sqrt{2/3}$, $y = \pm\sqrt{2/3}$. The four points are therefore $(\sqrt{2/3}, \sqrt{2/3})$, $(\sqrt{2/3}, -\sqrt{2/3})$, $(-\sqrt{2/3}, \sqrt{2/3})$, and $(-\sqrt{2/3}, -\sqrt{2/3})$.

The gray line meets the black ellipse where $x = 0$ and $0^2 + 2y^2 = 2$, so $y = \pm 1$. Similarly, the black line meets the gray ellipse where $y = 0$ and $x = \pm 1$. This yields the four points $(1, 0)$, $(-1, 0)$, $(0, 1)$, $(0, -1)$.

The final point, where the two straight lines meet, is $(0, 0)$.

14. b.



The figure above shows the contours of $C(x, y)$, but they are *not* evenly spaced. The contours shown are $C = -.25, 0, .75, 3$, and 8 . The contour $C = 0$ consists of the two intersecting ellipses. Their intersection points mark the four saddles. The contour $C = -.25$ consists of the four small ovals that appear on the diagonals. Each oval surrounds one of the four minima. The contour $C = .75$ has two parts: the nearly circular shape surrounding the origin and the closed curve just outside the pair of ellipses. The maximum is at the origin.

15. a. We have $E_x = \sin x = 0$ when $x = k\pi$, while $E_v = v = 0$ when $v = 0$. Therefore the critical points of $E(x, v)$ are $(x, v) = (k\pi, 0)$, for any integer k . Because $E(x, v)$ is a periodic function of x , with period 2π , we know all critical points with k will be the same, and all with k odd will be the same.

The graph of E appears in the text on page 516. In it we see that the critical point at the origin is a minimum while the critical point at $(\pi, 0)$ is a saddle.

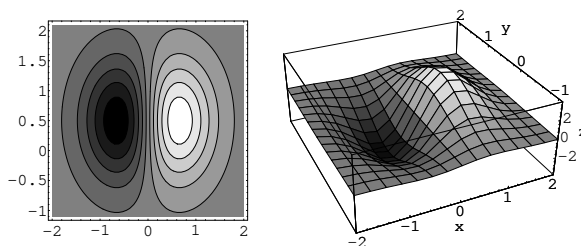
15. b. The critical points of E are exactly the equilibrium points of the associated dynamical system. Moreover, the minima of E are the attractors of the dynamical system, and the saddles of E are the saddles of the dynamical system.

16. In the method of steepest descent, we look for an attractor of the dynamical system

$$\frac{dx}{dt} = -\frac{\partial p}{\partial x} = -(-2x \sin x + \cos x)e^{2+y-x^2-y^2}$$

$$\frac{dy}{dt} = -\frac{\partial p}{\partial y} = -(1-2y) \sin x e^{2+y-x^2-y^2}$$

With a numerical differential equation solver we find a stable equilibrium at $x = -.653271\dots$, $y = .5$. It takes a bit of experimenting to find reasonable initial conditions. The graph of $p(x, y)$, shown below, indicates that $x = -1$, $y = 0$ is a good choice.



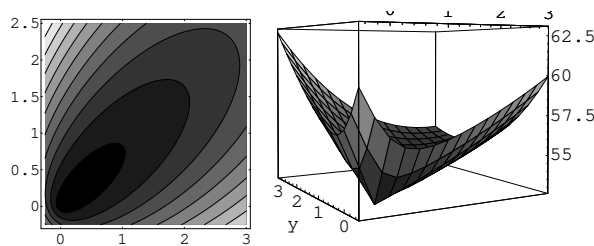
The Distribution Problem

17. a. The method of steepest descent leads us to look for attractors of the following dynamical system:

$$\frac{dx}{dt} = -\left(\frac{5x}{\sqrt{x^2+y^2}} + \frac{3(x-6)}{\sqrt{(x-6)^2+(y-9)^2}} + \frac{2(x-10)}{\sqrt{(x-10)^2+(y-2)^2}} \right)$$

$$\frac{dy}{dt} = -\left(\frac{5y}{\sqrt{x^2+y^2}} + \frac{3(y-9)}{\sqrt{(x-6)^2+(y-9)^2}} + \frac{2(y-2)}{\sqrt{(x-10)^2+(y-2)^2}} \right)$$

A numerical differential solver finds the attractor of this dynamical system at $(x, y) = (0, 0)$. This is a bit surprising, but it is consistent with the graph of $T(x, y)$. As the following figure shows, near $(0, 0)$, the graph of T looks like a cone whose vertex is at $(0, 0)$.

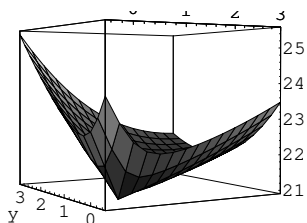


17. b. since there is much more activity at A , it makes sense for the distribution point to move towards A . It is perhaps more surprising that the distribution center should be exactly at A —but that's what our result says.

18. In the given circumstances, we must double the weight of the component of T connected with A : $T(x, y) = 2\sqrt{x^2 + y^2} + \sqrt{(x-6)^2 + (y-9)^2} + \sqrt{(x-10)^2 + (y-2)^2}$. Using the method of steepest descent, we look for attractors of the following dynamical system:

$$\begin{aligned}\frac{dx}{dt} &= - \left(\frac{2x}{\sqrt{x^2 + y^2}} + \frac{(x-6)}{\sqrt{(x-6)^2 + (y-9)^2}} \right. \\ &\quad \left. + \frac{(x-10)}{\sqrt{(x-10)^2 + (y-2)^2}} \right) \\ \frac{dy}{dt} &= - \left(\frac{2y}{\sqrt{x^2 + y^2}} + \frac{(y-9)}{\sqrt{(x-6)^2 + (y-9)^2}} \right. \\ &\quad \left. + \frac{(y-2)}{\sqrt{(x-10)^2 + (y-2)^2}} \right)\end{aligned}$$

With a numerical differential equation solver we find the attractor at $(x, y) = (0, 0)$. The distribution center should be at A . Once again the graph has locally the form of a cone with vertex at $(0, 0)$.



19. The function to minimize is

$$\begin{aligned}T(x, y) &= 32\sqrt{(x-200)^2 + (y-300)^2} \\ &\quad + 17\sqrt{(x-1920)^2 + (y-1100)^2} \\ &\quad + 20\sqrt{(x-2240)^2 + (y-450)^2} \\ &\quad + 41\sqrt{(x-2875)^2 + (y-1150)^2}\end{aligned}$$

We use the method of steepest descent and look for attrac-

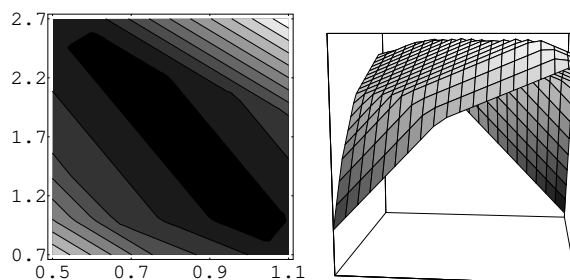
tors of the system.

$$\begin{aligned}\frac{dx}{dt} &= - \left(\frac{32(x-200)}{\sqrt{(x-200)^2 + (y-300)^2}} \right. \\ &\quad + \frac{17(x-1920)}{\sqrt{(x-1920)^2 + (y-1100)^2}} \\ &\quad + \frac{20(x-2240)}{\sqrt{(x-2240)^2 + (y-450)^2}} \\ &\quad \left. + \frac{41(x-2875)}{\sqrt{(x-2875)^2 + (y-1150)^2}} \right) \\ \frac{dy}{dt} &= - \left(\frac{32(y-300)}{\sqrt{(x-200)^2 + (y-300)^2}} \right. \\ &\quad + \frac{17(y-1100)}{\sqrt{(x-1920)^2 + (y-1100)^2}} \\ &\quad + \frac{20(y-450)}{\sqrt{(x-2240)^2 + (y-450)^2}} \\ &\quad \left. + \frac{41(y-1150)}{\sqrt{(x-2875)^2 + (y-1150)^2}} \right)\end{aligned}$$

With a numerical differential equation solver we find the attractor of this dynamical system is $(x, y) = (2149.6, 849.9)$.

The Best-Fitting Line

20. Features of the graph of $AE(m, b)$ are easier to see if you turn the graph upside down. That is done on the right, below. The graph now looks like a roof, and the minimum shows up as the ridge line of the roof. In other words, the minimum is achieved along an entire line segment. The line segment runs from about $(m, b) = (.67, 2.3)$ to $(m, b) = (1, 1)$. The contour plot shows the minimum as a line segment.



21. a. Two possible best-fitting lines are $Y_1 = .67x + 2.3$ and $Y_2 = x + 1$. If we use these choices, then $Y_1(5) = 5.65$

and $Y_2(5) = 6$. Other choices are possible; consequences of this are discussed in part b.

21. b. Start with the fact that all points (m, b) on the line segment from $(m, b) = (.67, 2.3)$ to $(m, b) = (1, 1)$ give “best-fitting” lines $Y = mx + b$. We can get an explicit formula for this line segment. We have $\Delta b / \Delta m = (1 - 2.3) / (1 - .67) = -130/33$; since $(1, 1)$ is a point on the segment, we can write

$$\frac{(b-1)}{(m-1)} = -\frac{130}{33} \quad \text{or} \quad b = -\frac{130}{33}m + \frac{163}{33}.$$

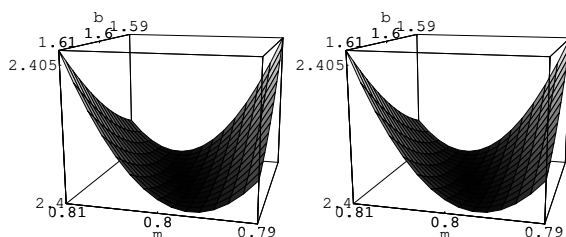
Therefore a best-fitting line has the form

$$Y = mx + -\frac{130}{33}m + \frac{163}{33} \quad \text{where } .67 \leq m \leq 1.$$

If $x = 5$ then $Y = (5 - \frac{130}{33})m + \frac{163}{33} = \frac{35}{33}m + \frac{163}{33}$. Therefore, when m is in the interval $[.67, 1]$, Y is in the interval $[(35 \times .67 + 163)/33, (35 \times 1 + 163)/33] = [5.65, 6]$.

$$\begin{aligned} \mathbf{22.} \quad SE(m, b) &= (1-b)^2 + (3-m-b)^2 + \\ &+ (4-2m-b)^2 + (3-3m-b)^2 + (5-4m-b)^2. \end{aligned}$$

23. a. The graph indicates a minimum at $(.8, 1.6)$ accurate to one decimal place. The contour plot also indicates a minimum at $(.8, 1.6)$, but this is evidently accurate to three decimal places.



23. b. With a little algebra we find

$$\frac{\partial SE}{\partial m} = 60m + 20b - 80 \quad \frac{\partial SE}{\partial b} = 20m + 10b - 32.$$

Therefore, the method of steepest descent analyzes

$$\frac{dm}{dt} = -(60m + 20b - 80) \quad \frac{db}{dt} = -(20m + 10b - 32).$$

Since this is a linear system, we can find its equilibrium exactly by solving a pair of linear equations. We find $(m, b) = (.8, 1.6)$. This method (at least in these circumstances) yields infinitely many decimal places accuracy. It is apparent from the graph of $SE(m, b)$ that this equilibrium is an attractor—and hence is a minimum of $SE(m, b)$.

Solutions: Chapter 10

Series and Approximations

10.1 Approximation Near a Point or Over an Interval

There are no exercises in this section.

10.2 Taylor Polynomials

1. a. Since $\sin(x) \approx x - x^3/3! + x^5/5! - x^7/7!$, we have $\sin(x)/x \approx 1 - x^2/3! + x^4/5! - x^6/7!$ and

$$\int \frac{\sin(x)}{x} dx \approx x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!}.$$

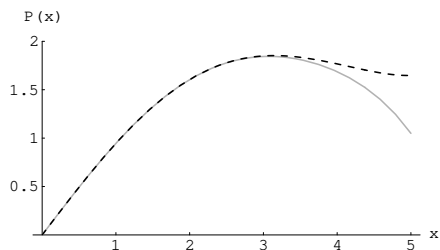
1. b. From page 601, $e^{x^2} \approx 1 + x^2 + x^4/2! + x^6/3!$, so

$$\int e^{x^2} dx \approx x + \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} + \frac{x^7}{7 \cdot 3!}.$$

1. c. Replacing x by x^2 in the formula for $\sin(x)$, we get $\sin(x^2) \approx x^2 - x^6/3!$, so

$$\int \sin(x^2) dx \approx \frac{x^3}{3} - \frac{x^7}{7 \cdot 3!}.$$

2.



The seventh-degree polynomial is shown in gray, the ninth-degree dashed. The two begin to differ visibly when $x \approx 3.5$.

3. The term added in the ninth-degree approximation to $E(t)$ is $+t^9/(9 \cdot 4!)$. Since this term is the first one that we

ignore when we use the seventh-degree approximation, it gives us some idea how big an error we make in using the seventh-degree approximation.

	$E(.3)$	$E(-1)$
7-th degree	.29123779	-.74285714
9th-degree	.29123788	-.74748677
accuracy	6 places	2 places

4.

	$\sin(.4)$	$\sin(\pi/12)$
polynomial	.3894183415	.2588190450
calculator	.3894183423	.2588190451
agreement	8 places	9 places

5. We construct the Taylor polynomial of $g(x)$ at $x = 1$ by evaluating the various derivatives of g at $x = 1$.

$$\begin{aligned} g(x) &= x^3 - 3x & g(1) &= -2 \\ g'(x) &= 3x^2 - 3 & g'(1) &= 0 \\ g''(x) &= 6x & g''(1) &= 6 \\ g'''(x) &= 6 & g'''(1) &= 6 \end{aligned}$$

Now, using the pattern described on pages 599–600, we build the Taylor polynomial:

$$P_3(x) = -2 + 3(x-1)^2 + (x-1)^3.$$

If we multiply this out, we get $P_3(x) = x^3 - 3x$, which is just the original function $g(x)$. Since $P_3(x) = g(x)$ for all x , no higher order terms are needed in order to improve the approximation: the *fourth*-degree Taylor polynomial ought to be the same as the third: $P_4(x) = P_3(x)$. Indeed, since $g^{(4)}(x) = 0$ for all x , the coefficient of $(x-1)^4$ in P_4 is zero.

6. a. Let $f(x) = \sin(x)$; then

$$\begin{aligned} f(x) &= f^{(4)}(x) = \sin(x) & f(\pi) &= f^{(4)}(\pi) = 0 \\ f'(x) &= f^{(5)}(x) = \cos(x) & f'(\pi) &= f^{(5)}(\pi) = -1 \\ f''(x) &= f^{(6)}(x) = -\sin(x) & f''(\pi) &= f^{(6)}(\pi) = 0 \\ f'''(x) &= f^{(7)}(x) = -\cos(x) & f'''(\pi) &= f^{(7)}(\pi) = 1 \end{aligned}$$

Therefore, when $x \approx \pi$,

$$\sin(x) \approx -(x - \pi) + \frac{(x - \pi)^3}{3!} - \frac{(x - \pi)^5}{5!} + \frac{(x - \pi)^7}{7!}.$$

6. b. Let $f(x) = \cos(x)$; then

$$\begin{aligned} f(x) &= f^{(4)}(x) = \cos(x) & f(\pi) &= f^{(4)}(\pi) = -1 \\ f'(x) &= f^{(5)}(x) = -\sin(x) & f'(\pi) &= f^{(5)}(\pi) = 0 \\ f''(x) &= f^{(6)}(x) = -\cos(x) & f''(\pi) &= f^{(6)}(\pi) = 1 \\ f'''(x) &= f^{(7)}(x) = \sin(x) & f'''(\pi) &= f^{(7)}(\pi) = 0 \end{aligned}$$

Therefore, when $x \approx \pi$,

$$\cos(x) \approx -1 + \frac{(x - \pi)^2}{2!} - \frac{(x - \pi)^4}{4!} + \frac{(x - \pi)^6}{6!}.$$

6. c. Let $f(x) = \sin(3x)$; then

$$\begin{aligned} f(x) &= \sin(3x) & f(\pi) &= 0 \\ f'(x) &= 3\cos(3x) & f'(\pi) &= -3 \\ f''(x) &= -3^2\sin(3x) & f''(\pi) &= 0 \\ f'''(x) &= -3^3\cos(3x) & f'''(\pi) &= 3^3 \\ f^{(4)}(x) &= 3^4\sin(3x) & f^{(4)}(\pi) &= 0 \\ f^{(5)}(x) &= 3^5\cos(3x) & f^{(5)}(\pi) &= -3^5 \\ f^{(6)}(x) &= -3^6\sin(3x) & f^{(6)}(\pi) &= 0 \\ f^{(7)}(x) &= -3^7\cos(3x) & f^{(7)}(\pi) &= 3^7 \end{aligned}$$

Therefore, when $x \approx \pi$, $\sin(3x) \approx$

$$-3(x - \pi) + \frac{3^3}{3!}(x - \pi)^3 - \frac{3^5}{5!}(x - \pi)^5 + \frac{3^7}{7!}(x - \pi)^7.$$

7. a. The seventh-degree polynomial Taylor polynomial centered at 0 gives $3 - 3^3/3! + 3^5/5! - 3^7/7! = .0910714$. If we go to the ninth degree, we must add $3^9/9! = .05424$. This is a relatively large number, and suggests that the seventh-degree approximation is very rough—with not even a single decimal place stabilized.

7. b. The seventh-degree polynomial Taylor polynomial centered at π gives $-(3 - \pi) + (3 - \pi)^3/3! - (3 - \pi)^5/5! + (3 - \pi)^7/7! = .141120008059804$. The ninth-degree term is $-(3 - \pi)^9/9! = -6.3 \times 10^{-14}$. This is extremely small, and suggests that the seventh-degree approximation is correct to 13 decimal places. In fact, $\sin(3) = .141120008059867\dots$, so this is indeed true.

8. The following is a QuickBasic program modeled on the one in the text on page 600, but it is simpler in several ways. The user sets the values of N and deg and

obtains the approximation as the value of Sum. Note that $\text{ATN}(1)$ is $\arctan(1) = \pi/4$. The tables following the program show values that have stabilized to seven decimal places. Notice that the degree n of the polynomial that is needed to obtain this stability must increase as the angle increases.

```
DEFDBL A-Z
N = 7
deg = 5
pi = 4 * ATN(1)
rad = deg * pi / 180
Sign = 1
Sum = 0
DEF fnfact(m)
  p = 1
  FOR r = 2 TO m
    p = p * r
  NEXT r
  fnfact = p
END DEF
FOR k = 1 TO N STEP 2
  Sum = Sum + Sign * rad ^ k / fnfact(k)
  Sign = -1 * Sign
NEXT k
PRINT N, Sum
```

n	$S_n(5^\circ)$	$S_n(10^\circ)$	$S_n(15^\circ)$
1	.087266463	.174532925	.261799388
3	.087155701	.173646829	.258808813
5	.087155743	.173648179	.258819062
7		.173648178	.258819045
n	$S_n(20^\circ)$	$S_n(25^\circ)$	$S_n(30^\circ)$
1	.349065850	.436332313	.523598776
3	.341977081	.422487061	.499674179
5	.342020268	.422618858	.500002133
7	.342020143	.422618260	.499999992
9	.342020143	.422618262	.500000000
n	$S_n(35^\circ)$	$S_n(40^\circ)$	$S_n(45^\circ)$
1	.610865238	.698131701	.785398163
3	.572873866	.641421547	.704652651
5	.573582702	.642803539	.707143046
7	.573576404	.642787502	.707106470
9	.573576436	.642787610	.707106783
11		.642787610	.707106781

9. a. If property A is to be true for 0 as well as positive integers, then

$$2^0 \cdot 2^n = 2^{0+n} = 2^n \quad \text{so} \quad 2^0 = \frac{2^n}{2^n} = 1.$$

9. b. If property A is to be true for negative as well as positive integers, then

$$2^{-3} \cdot 2^3 = 2^{-3+3} = 2^0 = 1 \quad \text{so} \quad 2^{-3} = \frac{1}{2^3}.$$

9. c. If property M is to be true for fractions as well as whole numbers, then

$$(2^{1/4})^4 = 2^{(1/4) \cdot 4} = 2^1 = 2, \text{ so } 2^{1/4} = \sqrt[4]{2}.$$

9. d. First of all, if n is a positive integer, then

$$(n+1)! = (n+1) \cdot \underbrace{n \cdot (n-1) \cdots 1}_{n!} = (n+1) \cdot n!,$$

so the property holds. If we want this property to hold for $n = 0$, then $1! = 1 \cdot 0!$. This will be true if we agree that $0! = 1$.

9. e. Now that $0!$ is defined, we can let $n = -1$ in $(n+1)! = (n+1) \cdot n!$ and get $1 = 0! = 0 \cdot (-1)!$. Now suppose $(-1)! = C$; then $0 \cdot C = 1$, but no number C exists for which this is true. Therefore it is impossible to give $(-1)!$ a value (or meaning) which will keep the property $(n+1)! = (n+1) \cdot n!$.

10. The values of the derivatives up to order 5 are:

$$\begin{aligned} f(x) &= (1+x)^{1/2} & f(0) &= 1 \\ f'(x) &= \frac{1}{2}(1+x)^{-1/2} & f'(0) &= \frac{1}{2} \\ f''(x) &= -\frac{1}{2} \cdot \frac{1}{2}(1+x)^{-3/2} & f''(0) &= -\frac{1}{4} \\ f'''(x) &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2}(1+x)^{-5/2} & f'''(0) &= \frac{1 \cdot 3}{8} \\ f^{(4)}(x) &= -\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}(1+x)^{-7/2} & f^{(4)}(0) &= -\frac{1 \cdot 3 \cdot 5}{16} \\ f^{(5)}(x) &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2} \cdot \frac{7}{2}(1+x)^{-9/2} & f^{(5)}(0) &= \frac{1 \cdot 3 \cdot 5 \cdot 7}{32} \end{aligned}$$

Notice the denominator of the n -th coefficient is 2^n . In building the Taylor polynomial we can therefore join the factor $1/2^n$ to x^n :

$$\begin{aligned} P_5(x) &= 1 + \frac{x}{2} - \frac{1}{2!} \left(\frac{x}{2}\right)^2 + \frac{1 \cdot 3}{3!} \left(\frac{x}{2}\right)^3 \\ &\quad - \frac{1 \cdot 3 \cdot 5}{4!} \left(\frac{x}{2}\right)^4 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{5!} \left(\frac{x}{2}\right)^5. \end{aligned}$$

Therefore $P_5(.1) = 1.048808867\dots$ while $\sqrt{1.1} = 1.048808848\dots$, an error of .0000000191. We see $P_5(1.1)$ is accurate to 7 decimal places.

11. This is very similar to the preceding exercise.

$$\begin{aligned} f(x) &= (1+x)^c \\ f'(x) &= c(1+x)^{c-1} \\ f''(x) &= c(c-1)(1+x)^{c-2} \\ f'''(x) &= c(c-1)(c-2)(1+x)^{c-3} \\ f^{(4)}(x) &= c(c-1)(c-2)(c-3)(1+x)^{c-4} \\ &\vdots \\ f^{(n)}(x) &= c(c-1)(c-2)\cdots(c-(n-1))(1+x)^{c-n} \end{aligned}$$

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= c \\ f''(0) &= c(c-1) \\ f'''(0) &= c(c-1)(c-2) \\ f^{(4)}(0) &= c(c-1)(c-2)(c-3) \\ &\vdots \\ f^{(n)}(0) &= c(c-1)(c-2)\cdots(c-n+1) \end{aligned}$$

$$\begin{aligned} P_n(x) &= 1 + cx + \frac{c(c-1)}{2!}x^2 + \frac{c(c-1)(c-2)}{3!}x^3 + \\ &\quad \cdots + \frac{c(c-1)(c-2)\cdots(c-n+1)}{n!}x^n \end{aligned}$$

12. We can use the previous exercise by taking $c = -1/3$ and replacing x by x^2 :

$$\begin{aligned} \frac{1}{\sqrt[3]{1+x^2}} &= (1+x^2)^{-1/3} \\ &\approx 1 - \frac{1}{3}x^2 + \frac{(-1/3)(-4/3)}{2!}(x^2)^2 + \frac{(-1/3)(-4/3)(-7/3)}{3!}(x^2)^3 \\ &= 1 - \frac{1}{3}x^2 + \frac{2}{9}x^4 - \frac{14}{81}x^6 \end{aligned}$$

13. We use the Taylor polynomial approximation from the previous exercise.

$$\begin{aligned} \int_0^1 \frac{1}{\sqrt[3]{1+x^2}} dx &\approx \int_0^1 \left(1 - \frac{x^2}{3} + \frac{2x^4}{9} - \frac{14x^6}{81}\right) dx \\ &= x - \frac{x^3}{3 \cdot 3} + \frac{2x^5}{5 \cdot 9} - \frac{14x^7}{7 \cdot 81} \Big|_0^1 \\ &= \frac{368}{405} \approx .9086419753. \end{aligned}$$

14. Our starting point is the Taylor polynomial approximation from page 612 of the text: $\operatorname{erf}(t) \approx$

$$\frac{2}{\sqrt{\pi}} \left(t - \frac{t^3}{3} + \frac{t^5}{5 \cdot 2!} - \frac{t^7}{7 \cdot 3!} + \cdots \pm \frac{t^{2n+1}}{(2n+1) \cdot n!} \right).$$

The ninth-degree approximation gives $\operatorname{erf}(.3) \approx .3286267609\dots$. Since the *eleventh*-degree approximation is $\operatorname{erf}(.3) = .3286267594\dots$, and this agrees with the ninth-degree approximation in the first seven decimal places, we can consider those seven places to be correct and we can consider that the ninth-degree approximation found them.

15. a. Some of these calculations get quite lengthy; a symbolic differentiation program is useful here.

$$\begin{aligned}f(x) &= \tan(x) \\f'(x) &= \sec^2(x) \\f''(x) &= 2\sec^2(x)\tan(x) \\f^{(3)}(x) &= 2\sec^4(x) + 4\sec^2(x)\tan^2(x) \\f^{(4)}(x) &= 16\sec^4(x)\tan(x) + 8\sec^2(x)\tan^3(x) \\f^{(5)}(x) &= 16\sec^6(x) + 88\sec^4(x)\tan^2(x) \\&\quad + 16\sec^2(x)\tan^4(x)\end{aligned}$$

Therefore

$$\begin{aligned}f(0) &= 0 & f'(0) &= 1 & f''(0) &= 0 \\f^{(3)}(0) &= 2 & f^{(4)}(0) &= 0 & f^{(5)}(0) &= 16\end{aligned}$$

and finally

$$\tan(x) \approx x + \frac{2}{3!}x^3 + \frac{16}{5!}x^5 = x + \frac{1}{3}x^3 + \frac{2}{15}x^5$$

15. b. We want the fact $\cos(x) \cdot \tan(x) = \sin(x)$ to be reflected in the Taylor polynomials for these three functions. We have

$$\begin{aligned}\cos(x)\tan(x) &\approx \\&\left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)\left(x + \frac{x^3}{3} + \frac{2x^5}{15}\right) \\&= x + \frac{x^3}{3} + \frac{2x^5}{15} - \frac{x^3}{2} - \frac{x^5}{6} + \frac{x^5}{24} + R(x) \\&= x + \left(\frac{1}{3} - \frac{1}{2}\right)x^3 + \left(\frac{2}{15} - \frac{1}{6} + \frac{1}{24}\right)x^5 + R(x) \\&= x - \frac{x^3}{6} + \frac{x^5}{120} + R(x).\end{aligned}$$

Here $R(x)$ consists of terms involving powers of x higher than x^5 . Since the polynomial accompanying $R(x)$ in the last line is the fifth-degree Taylor polynomial for $\sin(x)$, we have shown what was asked.

16. To make the derivatives easier to calculate, write $f(x) = 1/(1-x)$ as $(1-x)^{-1}$. Then we have the following.

$$\begin{aligned}f(x) &= (1-x)^{-1} & f(0) &= 1 \\f'(x) &= -1 \cdot (-1) \cdot (1-x)^{-2} & f'(0) &= 1 \\f''(x) &= 1 \cdot 2 \cdot (1-x)^{-3} & f''(0) &= 2 \\f'''(x) &= 1 \cdot 2 \cdot 3 \cdot (1-x)^{-4} & f'''(0) &= 3! \\f^{(4)}(x) &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot (1-x)^{-5} & f^{(4)}(0) &= 4!\end{aligned}$$

$$\begin{aligned}f^{(5)}(x) &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot (1-x)^{-6} & f^{(5)}(0) &= 5! \\&\vdots & &\vdots \\f^{(n)}(x) &= 1 \cdot 2 \cdot 3 \cdots n \cdot (1-x)^{-n-1} & f^{(n)}(0) &= n!\end{aligned}$$

Therefore, by the rules for constructing a Taylor polynomial,

$$\begin{aligned}\frac{1}{1-x} &\approx 1 + x + \frac{2x^2}{2!} + \frac{3!x^3}{3!} + \cdots + \frac{n!x^n}{n!} \\&= 1 + x + x^2 + x^3 + \cdots + x^n\end{aligned}$$

$$\begin{aligned}\mathbf{17.} \quad \ln(1-x) &= -\int \frac{1}{1-x} dx \\&\approx -\int 1 + x + x^2 + \cdots + x^n dx \\&= -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^{n+1}}{n+1}\right).\end{aligned}$$

18. a. We first construct the Taylor polynomial directly from the derivatives of $f(x) = \ln(x)$, evaluated at $x = 1$.

$$\begin{aligned}f(x) &= \ln(x) & f(1) &= 0 \\f'(x) &= x^{-1} & f'(1) &= 1 \\f''(x) &= -x^{-2} & f''(1) &= -1 \\f'''(x) &= 2x^{-3} & f'''(1) &= 2 \\f^{(4)}(x) &= -2 \cdot 3x^{-4} & f^{(4)}(1) &= -3! \\f^{(5)}(x) &= 2 \cdot 3 \cdot 4x^{-5} & f^{(5)}(1) &= 4! \\&\vdots & &\vdots\end{aligned}$$

In general, $f^{(n)}(x) = \pm 2 \cdot 3 \cdots (n-1)x^{-n}$, and the minus sign should be chosen precisely when n is even. Since the expression $(-1)^{n+1}$ is negative precisely when n is an even integer, we can write $f^{(n)}(1) = (-1)^{n+1}(n-1)!$. Therefore

$$\begin{aligned}\ln(x) &\approx (x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \cdots \\&\quad \cdots (-1)^{n+1} \frac{(n-1)!}{n!} (x-1)^n \\&= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \cdots \\&\quad \cdots (-1)^{n+1} \frac{(x-1)^n}{n}.\end{aligned}$$

18. b. To use exercise 17, note that if x is near 0, then $u = 1-x$ is near 1. This suggests that a Taylor polynomial for $\ln(1-x)$ centered at $x = 0$ can be converted into a

Taylor polynomial for $\ln(u)$ centered at 1 by making the substitution $u = 1 - x$. Thus (since $x = 1 - u$),

$$\ln(1-x) \approx -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^{n+1}}{n+1}\right)$$

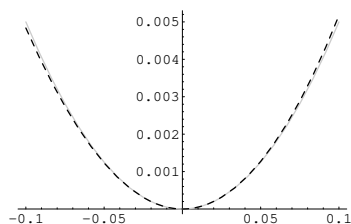
becomes

$$\begin{aligned} \ln(u) &\approx -\left((1-u) + \frac{(1-u)^2}{2} + \frac{(1-u)^3}{3} + \cdots \right. \\ &\quad \left. \cdots + \frac{(1-u)^{n+1}}{n+1}\right) \\ &= (u-1) - \frac{(u-1)^2}{2} + \frac{(u-1)^3}{3} - \cdots \\ &\quad \cdots (-1)^{n+1} \frac{(u-1)^{n+1}}{n+1} \end{aligned}$$

Note that we converted every factor $(1-u)^k$ into $(-1)^k(u-1)^k$, and then multiplied every term by the minus sign that was outside the parentheses, turning $-(1-u)^k$ into $(-1)^{k+1}(u-1)^k$.

If we replace u by x in this last expression for $\ln(u)$, we recover the statement made in part a about the Taylor polynomial of $\ln(x)$ centered at $x = 1$. Thus the Taylor polynomial centered at 0, which was found in the previous exercise, is consistent with the Taylor polynomial centered at 1, found in part a of this exercise.

19. a.

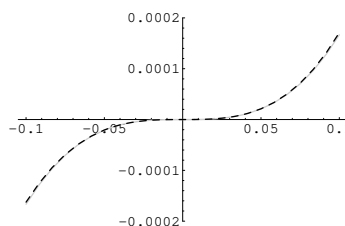


The figure above, in which $y = R_1(x)$ is plotted with dashes, suggests that the graph of $y = R_1(x)$ is close to a parabola, $R_1(x) \approx C_2 x^2$ as $x \rightarrow 0$, implying thereby that $R_1(x) = O(x^2)$ as $x \rightarrow 0$.

19. b. The figure above, in which $y = x^2/2$ is plotted as a solid gray curve, makes the case that $C_2 = 1/2$. If there is indeed a C_2 for which $R_1(x) \approx C_2 x^2$ as $x \rightarrow 0$, then $R_1(x)/x^2 \rightarrow C_2$ as $x \rightarrow 0$. Therefore, we can show that there is a C_2 , and determine its value, by evaluating $R_1(x)/x^2$ for values of x approaching 0. The following table shows $C_2 = .5 = 1/2$.

x	$R_1(x)/x^2$
10^{-1}	.517092
10^{-2}	.501671
10^{-3}	.500167
10^{-4}	.500017
10^{-5}	.500002
10^{-6}	.500000

20. a. The figure below, in which $y = R_2(x)$ is plotted with dashes, suggests that $R_2(x)$ is close to a cubic, $R_2(x) \approx C_3 x^3$ as $x \rightarrow 0$, implying thereby that $R_2(x) = O(x^3)$ as $x \rightarrow 0$.



20. b. The figure above, in which $y = x^3/6$ is plotted as a solid gray curve, makes the case that $C_3 = 1/6$. We can show that there is a C_3 , and determine its value, by evaluating $R_2(x)/x^3$ for values of x approaching 0. The following table shows $C_3 = .1666\ldots = 1/6$.

x	$R_2(x)/x^3$
10^{-1}	.170918
10^{-2}	.167084
10^{-3}	.166708
10^{-4}	.166671
10^{-5}	.166667

21. R_3 is defined using $P_3(x) = 1 + x + x^2/2 + x^3/6$. We can show $R_3(x) = O(x^4)$ by showing that $R_3(x)/x^4$ converges as x approaches 0.

x	$R_3(x)/x^4$
10^{-1}	.00425
10^{-2}	.00418
10^{-3}	.00417
10^{-4}	.00417

(QuickBasic has difficulty getting more precision than this.) It appears that $R_3(x)/x^4$ converges to $C_4 = .0041666\ldots = 1/24$.

22. There is no conflict; Taylor's theorem is not wrong here. It simply happens that, in this case, $O(x^4) = 0 \cdot x^4 + O(x^5)$ as $x \rightarrow 0$. In other words, if $R(x) = O(x^5)$ as $x \rightarrow 0$, then $R(x) = O(x^4)$ as $x \rightarrow 0$. To see this is true, we must show $R(x)/x^4$ has a limit as $x \rightarrow 0$, given that $R(x)/x^5$ has a limit—let us call it L :

$$\lim_{x \rightarrow 0} \frac{R(x)}{x^4} = \lim_{x \rightarrow 0} x \cdot \frac{R(x)}{x^5} = 0 \cdot L = 0.$$

The limit is 0, so $R(x) = O(x^4)$.

23. a.
$$\frac{\sin(x)}{x} = \frac{x - x^3/6 + O(x^5)}{x} = 1 - x^2/6 + O(x^4) \rightarrow 1 \text{ as } x \rightarrow 0.$$

$$\begin{aligned} 23. \text{ b. } \frac{e^x - (1+x)}{x^2} &= \frac{1+x+x^2/2+O(x^3) - (1+x)}{x^2} \\ &= 1/2 + O(x) \rightarrow 1/2 \text{ as } x \rightarrow 0. \end{aligned}$$

$$\begin{aligned} 23. \text{ c. } \frac{\ln(x)}{x-1} &= \frac{(x-1) - (x-1)^2/2 + O((x-1)^3)}{x-1} \\ &= 1 - (x-1)/2 + O((x-1)^2) \rightarrow 1 \text{ as } x \rightarrow 1. \end{aligned}$$

$$\begin{aligned} 23. \text{ d. } \frac{x - \sin(x)}{x^3} &= \frac{x - (x - x^3/6 + O(x^5))}{x^3} \\ &= 1/6 + O(x^2) \rightarrow 1/6 \text{ as } x \rightarrow 0. \end{aligned}$$

$$\begin{aligned} 23. \text{ e. } \frac{\sin(x^2)}{1 - \cos(x)} &= \frac{x^2 - x^6/6 + O(x^{10})}{1 - (1 - x^2/2 + O(x^4))} \\ &= \frac{1 - x^4/6 + O(x^8)}{1/2 + O(x^2)} \rightarrow \frac{1}{1/2} = 2 \text{ as } x \rightarrow 0. \end{aligned}$$

24. In the formula $f(x) = 1 + x + O(x^4)$, the expression $O(x^4)$ stands for a particular function, but one that is not explicitly identified. Therefore, we can carry out algebraic operations that involve it.

$$\begin{aligned} (f(x))^2 &= (1 + x + O(x^4))(1 + x + O(x^4)) \\ &= 1 + x + O(x^4) + x(1 + x + O(x^4)) \\ &\quad + O(x^4)(1 + x + O(x^4)) \\ &= 1 + 2x + x^2 + \underbrace{O(x^4)(2 + 2x + O(x^4))}_{O(x^4)} \end{aligned}$$

as $x \rightarrow 0$. Note that the terms gathered together form a new $O(x^4)$, so we can write $(f(x))^2 = 1 + 2x + x^2 + O(x^4)$.

25. a. This is similar to the previous exercise.

$$\begin{aligned} (\sin(x))^2 &= (x - x^3/6 + O(x^5))(x - x^3/6 + O(x^5)) \\ &= x^2 - 2x^4/6 \\ &\quad + \underbrace{x^6/36 + O(x^5)(2x - 2x^3/6 + O(x^5))}_{O(x^6)} \\ &= x^2 - x^4/3 + O(x^6) \quad (x \rightarrow 0). \end{aligned}$$

The key step is to note that many terms are gathered up into a new function of order 6, as indicated by the brace.

25. b. We can do this as a product of polynomials:

$$\begin{array}{rcccccl} 1 & - & \frac{1}{2}x^2 & + & \frac{1}{24}x^4 & + & O(x^5) \\ 1 & - & \frac{1}{2}x^2 & + & \frac{1}{24}x^4 & + & O(x^5) \\ \hline 1 & - & \frac{1}{2}x^2 & + & \frac{1}{24}x^4 & + & O(x^5) \\ & - & \frac{1}{2}x^2 & + & \frac{1}{4}x^4 & + & O(x^6) \\ & & & + & \frac{1}{24}x^4 & + & O(x^6) \\ & & & & & + & O(x^5) \\ \hline 1 & - & x^2 & + & \frac{1}{3}x^4 & + & O(x^5) \end{array}$$

In this calculation, $O(x^n)$ ($n = 5$ or 6) appears several times; typically, it stands for *different* functions of order n in its different occurrences.

25. c. When we add the expressions for $(\sin x)^2$ and $(\cos x)^2$, the terms of degree 2 and 4 cancel, leaving only

$$(\sin x)^2 + (\cos x)^2 = 1 + O(x^5) + O(x^6) = 1 + O(x^5)$$

as $x \rightarrow 0$. We combine $O(x^5)$ and $O(x^6)$ because $O(x^6)$ is also of order 5. This point is addressed in the solution to exercise 22.

26. a. This is identical to exercise 15 a.

26. b. This is essentially the same as exercise 15 b. The function $R(x)$ introduced there satisfies $R(x) = O(x^5)$ as $x \rightarrow 0$.

27. a. Let $u = \sin(x) = x - x^3/6 + O(x^5)$ as $x \rightarrow 0$. Then

$$u^2 = x^2 + O(x^4) \quad u^3 = x^3 + O(x^6) \quad u^4 = O(x^4)$$

as $x \rightarrow 0$. Therefore $O(u^4) = O(x^4)$ as $x \rightarrow 0$ and

$$\begin{aligned} e^u &= 1 + u + \frac{1}{2}u^2 + \frac{1}{6}u^3 + O(u^4) \\ &= 1 + x - x^3/6 + O(x^5) + \frac{1}{2}(x^2 + O(x^4)) \\ &\quad + \frac{1}{6}(x^3 + O(x^6)) + O(x^4) \\ &= 1 + x + \frac{1}{2}x^2 + 0 \cdot x^3 + O(x^4) \quad (x \rightarrow 0). \end{aligned}$$

The point to note is that the cubic terms cancel completely.

27. b. We must calculate derivatives of $f(x) = e^{\sin(x)}$ and then evaluate them at $x = 0$.

$$\begin{aligned} f(x) &= e^{\sin(x)} \\ f'(x) &= \cos(x)e^{\sin(x)} \\ f''(x) &= (\cos^2(x) - \sin(x))e^{\sin(x)} \\ f'''(x) &= (\cos^3(x) - 3\cos(x)\sin(x) - \cos(x))e^{\sin(x)} \end{aligned}$$

Thus

$$f(0) = 1 \quad f'(0) = 1 \quad f''(0) = 1 \quad f'''(0) = 0,$$

so the cubic term is missing while the lower-order ones have simple forms. We thereby re-establish the formula we already found in part a:

$$f(x) = e^{\sin(x)} = 1 + x + \frac{1}{2}x^2 + O(x^4) \quad (x \rightarrow 0).$$

28. Using the formula for $(e^x - 1)/x$ given in the text, we have

$$\begin{aligned} \frac{x}{e^x - 1} &= \frac{1}{1 + \frac{1}{2}x + \frac{1}{6}x^2 + \frac{1}{24}x^3 + O(x^4)} \\ &= \frac{1}{1 + A} = 1 - A + A^2 - A^3 + O(A^4), \end{aligned}$$

where $A = \frac{1}{2}x + \frac{1}{6}x^2 + \frac{1}{24}x^3 + O(x^4)$. To compute $1 - A + A^2 - A^3 + O(A^4)$, we first calculate:

$$\begin{aligned} A &= \frac{x}{2} \left(1 + \frac{x}{3} + \frac{x^2}{12} + O(x^3) \right), \\ A^2 &= \frac{x^2}{4} \left(1 + \frac{2x}{3} + O(x^2) \right), \\ A^3 &= \frac{x^3}{8} (1 + O(x)), \\ A^4 &= O(x^4), \end{aligned}$$

It follows that

$$\begin{aligned} 1 - A &= 1 - \frac{1}{2}x - \frac{1}{6}x^2 - \frac{1}{24}x^3 + O(x^4), \\ A^2 &= \frac{1}{4}x^2 + \frac{1}{6}x^3 + O(x^4), \\ -A^3 &= -\frac{1}{8}x^3 + O(x^4), \\ O(A^4) &= O(x^4), \\ \text{so } \frac{x}{e^x - 1} &= 1 - \frac{1}{2}x + \frac{1}{12}x^2 + O(x^4). \end{aligned}$$

29. To show $f(x) = O(g(x))$ as $x \rightarrow a$, we must show that $f(x)/g(x)$ has a finite limit as $x \rightarrow a$.

29. a. $\frac{x+1/x}{x} = 1 + 1/x^2 \rightarrow 1$ as $x \rightarrow \infty$,

so $x + 1/x = O(x)$ as $x \rightarrow \infty$.

29. b. $\frac{5x^7 - 12x^4 + 9}{x^7} = 5 - 12/x^3 + 9/x^7 \rightarrow 5$
as $x \rightarrow \infty$.

29. c. $\frac{\sqrt{1+x^2}}{x} = \sqrt{\frac{1+x^2}{x^2}} = \sqrt{\frac{1}{x^2} + 1} \rightarrow \sqrt{1} = 1$
as $x \rightarrow \infty$.

29. d. $\frac{\sqrt{1+x^p}}{x^{p/2}} = \sqrt{\frac{1+x^p}{x^p}} = \sqrt{\frac{1}{x^p} + 1} \rightarrow \sqrt{1} = 1$
as $x \rightarrow \infty$. Note that we must have $p > 0$ for this to work.

30. a. Since $f(x) = \ln(x)$, $f^{(4)}(x) = -6/x^4$ and $|f^{(4)}(x)| = 6/x^4$. The set $|x-1| \leq .5$ is just the interval $.5 \leq x \leq 1.5$, and $x^{-4} = 1/x^4$ assumes its largest value at the *left* end of this interval: $x = .5$. Therefore

$$|f^{(4)}(x)| \leq 6 \cdot (.5)^{-4} = 6 \cdot 2^4 = 96.$$

We can take $M = 96$; no smaller number will work.

30. b. According to Taylor's theorem, version 3 (page 608),

$$|R_3(x)| \leq \frac{96}{4!} |x-1|^4 = 4|x-1|^4,$$

for all $|x-1| \leq .5$. Thus we can take $K = 4$.

30. c. In the interval $.5 \leq x \leq 1.5$ we have $|x-1| \leq \frac{1}{2}$ so the inequality in the previous part implies

$$|R_3(x)| \leq 4 \left(\frac{1}{2} \right)^4 = \frac{1}{4} = .25 \times 10^0.$$

By what was said on page 605, we see that $P_3(x) = \ln(x)$ to *zero* decimal places of accuracy. In other words, when you use $P_3(x)$ to approximate $\ln(x)$ in the interval $.5 \leq x \leq 1.5$, you can be sure only that the digits *before* the decimal point in the approximation are correct.

This conclusion is overly conservative (though all that can be supported on the given grounds). The agreement between $P_3(x)$ and $\ln(x)$ is much better than we can infer here. The worst case happens at $x = .5$, where $P_3(.5) = -.6666\dots$ while $\ln(.5) = -.6931\dots$; the first decimal place is correct.

30. d. If we restrict the interval to $|x-1| \leq .1$, then M is found by evaluating $6/x^4$ at $x = .9$: $M = 6/(.9)^4 = 9.14495$. It then follows that K is $9.14495/4! = .381039$ and

$$|R_3(x)| \leq .38109 \times (.1)^4 < .5 \times 10^{-4}$$

on the interval $.9 \leq x \leq 1.1$. Now we know that the first *four* decimal places in the approximation are correct.

31. We are given that $\lim_{x \rightarrow 0} \phi(x)/x^6 = L$, for some L .

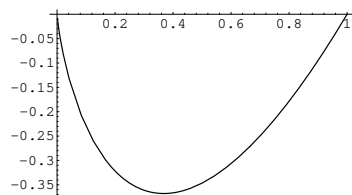
31. a. $\lim_{x \rightarrow 0} \phi(x)/x^5 = \lim_{x \rightarrow 0} x \cdot \phi(x)/x^6 = 0 \cdot L = 0$, so $\phi(x) = O(x^5)$ as $x \rightarrow 0$.

31. b. The argument in part a *also* shows $\phi(x) = o(x^5)$ as $x \rightarrow 0$.

31. c. Take $\phi(x) = x^6$. (Many other choices are possible; this is just a simple one.) Then $\phi(x) = O(x^6)$ as $x \rightarrow 0$. However, $\phi(x)/x^7 = 1/x$, and this does *not* have a finite limit as $x \rightarrow 0$. Therefore it is false that $\phi(x) = O(x^7)$ (even though $\phi(x) = O(x^6)$) as $x \rightarrow 0$.

31. d. This is similar to the previous part. Again take $\phi(x) = x^6$. If it were true that $\phi(x) = o(x^6)$ as $x \rightarrow 0$, then $\lim_{x \rightarrow 0} \phi(x)/x^6$ would be 0. But $\phi(x)/x^6 \equiv 1$, so the limit exists, but it is 1, *not* 0. Therefore it is false that $\phi(x) = o(x^6)$ as $x \rightarrow 0$.

32.



Shown above is the graph of $x \ln(x)$ over $0 < x \leq 1$. Notice that $x \ln(x) \rightarrow 0$ as $x \rightarrow 0$. But then

$$\lim_{x \rightarrow 0} \frac{\ln(x)}{1/x} = \lim_{x \rightarrow 0} x \ln(x) = 0,$$

which shows that $\ln(x) = o(1/x)$ as $x \rightarrow 0$.

10.3 Taylor Series

1. a. QuickBasic can't handle numbers as large as the ones that appear here. Something like Mathematica is needed. Let $f(n) = \frac{100^n}{n!}$; by trial and error we find $f(268) \approx 1.090$ while $f(269) \approx .4053$, so we need $n \geq 269$.

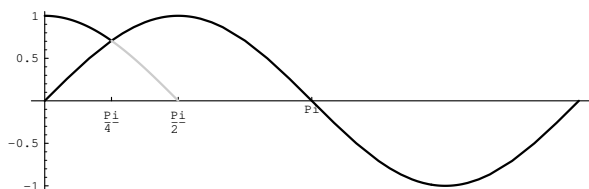
1. b. Because the sine function is periodic with period 2π , we have $\sin(100) = \sin(100 - 2k\pi)$ for any integer k . Since $100/2\pi = 15.9155$, if we choose $k = 16$ then $100 - 2k\pi$ is small: $100 - 2k\pi = -.530965$. In particular, we have $\sin(100) = \sin(100 - 32\pi)$, so we can evaluate the Taylor series at $100 - 32\pi$.

This is much quicker. For example, the ninth-degree term is approximately $(.530965)^9/9! = 1.076 \times 10^{-8} < .5 \times 10^{-7}$. This indicates that if we cut off the Taylor series after the seventh-degree term, we'll get a value for $\sin(100)$ that is accurate to seven decimal places. In fact,

$$\begin{aligned} P_7(100 - 32\pi) &= -.50636563189 \\ \sin(100) &= -.50636564111 \end{aligned}$$

so the first seven decimal places are indeed correct.

1. c. The basic idea is that any portion of the sine graph is a translation or a reflection of the portion of the sine graph between 0 and $\pi/2$, and that the second half of *that* graph is a reflection of the portion of the cosine graph between 0 and $\pi/4$.



In what follows, we show how to get the value of $\sin(x)$ for a larger and larger range of x values, starting with $x \in [0, \frac{\pi}{4}]$.

- If $x \in [0, \frac{\pi}{4}]$, just use $\sin(x)$.
- If $x \in [\frac{\pi}{4}, \frac{\pi}{2}]$, then $\frac{\pi}{2} - x \in [0, \frac{\pi}{4}]$; use $\cos(\frac{\pi}{2} - x)$.

At this stage, $\sin(x)$ is now determined for all $x \in [0, \frac{\pi}{2}]$ using the sine and cosine functions only on $[0, \frac{\pi}{4}]$.

- If $x \in [\frac{\pi}{2}, \pi]$, then $\pi - x \in [0, \frac{\pi}{2}]$; use $\sin(\pi - x)$.

At this stage, $\sin(x)$ is now determined for all $x \in [0, \pi]$ using the sine and cosine functions only on $[0, \frac{\pi}{4}]$.

- If $x \in [\pi, 2\pi]$, then $2\pi - x \in [0, \pi]$; use $-\sin(2\pi - x)$.

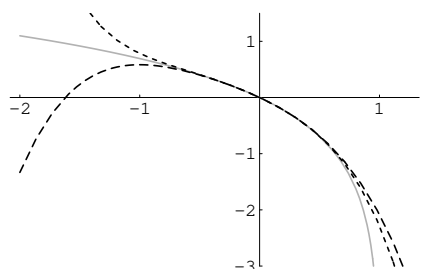
At this stage, $\sin(x)$ is now determined for all $x \in [0, 2\pi]$ using the sine and cosine functions only on $[0, \frac{\pi}{4}]$.

- If $x \in [2k\pi, 2(k+1)\pi]$, then $x - 2k\pi \in [0, 2\pi]$; use $\sin(x - 2k\pi)$.

At this stage, $\sin(x)$ is now determined for *all* x using the sine and cosine functions only on $[0, \frac{\pi}{4}]$.

2. a. When $x = -4$, the successive Taylor polynomials fluctuate between negative and positive values which grow increasingly in size with no limit. This happens because the individual terms, which are of the form $\pm 4^n/n$, blow up.

Students should experiment and see that the Taylor series behaves well when $-1 < x < 1$. (Students may see that the series is also well-behaved when $x = -1$, but that is not what is expected here.) One form of evidence for the interval of convergence is a graph. Here are graphs of $\ln(1-x)$ (gray) together with $P_4(x)$ (long dashes) and $P_6(x)$ (short dashes).



2. b. Since $\ln(1/A) = -\ln(A)$, $\ln(5) = -\ln(1/5)$. Now use the Taylor series for $\ln(1-x)$ to find $\ln(1/5)$; this

means we want $1 - x = 1/5$, so $x = 4/5$ and

$$\ln 5 = \sum_{k=1}^{\infty} \frac{(4/5)^k}{k}.$$

(Note the minus sign is gone because $\ln(1/A) = -\ln(A)$.) This series converges slowly, so we calculate partial sums using a program:

```
DEFDBL A-Z
N = 10
x = 4 / 5
Sum = 0
FOR k = 1 TO N
    Sum = Sum + x ^ k / k
NEXT k
PRINT N, Sum
```

N	$P_N(4/5)$
10	1.57887466675
20	1.60754586712
30	1.60929450057
40	1.60942599653
50	1.60943686840
60	1.60943781773
65	1.60943788363
70	1.60943790363
75	1.60943790973
76	1.60943791030
77	1.60943791075
78	1.60943791110
79	1.60943791138
80	1.60943791160

Thus, $\ln(5) = 1.6094379$ to seven decimal places. The Taylor polynomial of degree 70 gave us this value, but we had to calculate several more terms to be sure the value was stable.

2. c. It is better to use $x = 1/3$. The series converges faster for $x = 1/3$ than it does for $x = -1/2$, because $1/3$ is smaller in absolute value, so its powers (which appear in the Taylor series) get smaller faster.

In fact, when $x = -1/2$, the 5th degree Taylor polynomial gives the value $-.4072916\dots$, which is $\ln(1.5)$ accurate to two decimal places. When $x = 1/3$, the 5th degree Taylor polynomial gives the value $-.4051440\dots$, which is $\ln(1.5)$ accurate to *three* decimal places.

3. a. Since we know the Taylor series for $\ln(1-x)$, if we make the substitution $-x = u$ (or $x = -u$) we will get the Taylor series for $\ln(1+u)$:

$$\begin{aligned}\ln(1+u) &= -\left((-u) + \frac{(-u)^2}{2} + \frac{(-u)^3}{3} \dots\right) \\ &= u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \dots\end{aligned}$$

3. b. We use $\ln(A/B) = \ln(A) - \ln(B)$:

$$\begin{aligned}\ln\left(\frac{1-u}{1+u}\right) &= \ln(1-u) - \ln(1+u) \\ &= -u - \frac{u^2}{2} - \frac{u^3}{3} - \frac{u^4}{4} - \dots \\ &\quad - \left(u - \frac{u^2}{2} + \frac{u^3}{3} - \frac{u^4}{4} + \dots\right) \\ &= -2\left(u + \frac{u^3}{3} + \frac{u^5}{5} + \dots\right)\end{aligned}$$

3. c. To show that any $x > 0$ can be written in the form $(1-u)/(1+u)$ for some u in the interval $-1 < u < 1$, we can solve the equation $x = (1-u)/(1+u)$ for u and then show that $-1 < u < 1$. We have

$$x + xu = 1 - u \Rightarrow u + xu = 1 - x \Rightarrow u = \frac{1-x}{1+x},$$

so we can always determine u from x . It remains to show $-1 < (1-x)/(1+x) < 1$. Here is one side:

$$-1 < 1 \Rightarrow -(1+x) = -1-x < 1-x \Rightarrow -1 < \frac{1-x}{1+x},$$

and here is the other:

$$-x < +x \Rightarrow 1-x < 1+x \Rightarrow \frac{1-x}{1+x} < 1.$$

3. d. If $x = 5$, then $u = (1-5)/(1+5) = -2/3$. We use the Taylor series obtained in part b to approximate $\ln(5)$:

$$\begin{aligned}2\left(2/3 + \frac{(2/3)^3}{3} + \frac{(2/3)^5}{5} + \dots + \frac{(2/3)^N}{N}\right) \\ = 2 \sum_{n=0}^N \frac{(2/3)^{2n+1}}{2n+1}.\end{aligned}$$

The following program will compute this sum.

```
DEFDBL A-Z
N = 35
x = 2 / 3
Sum = 0
FOR k = 1 TO N STEP 2
    Sum = Sum + 2 * x ^ k / k
NEXT k
PRINT N, Sum
```

N	$P_N(2/3)$
1	1.3333333333
7	1.60026128421
13	1.60893429490
19	1.60940567311
25	1.60943568340
31	1.60943775098
33	1.60943784462
35	1.60943788387
37	1.60943790037
39	1.60943790732
41	1.60943791027
33	1.60943791151

This time we get $\ln(5)$ to seven decimal places with a polynomial of degree 37—roughly about half as far as in exercise 2.

4. a. The Taylor series for $\arctan(x)$ can be found on page 603:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots = \sum_{n=0}^N (-1)^n \frac{x^{2n+1}}{2n+1}$$

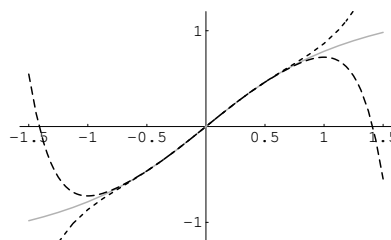
The following program will calculate estimates of $\arctan(.5)$ from the series:

```
DEFDBL A-Z
N = 23
x = .5
Sign = -1
Sum = x
FOR k = 3 TO N STEP 2
    Sum = Sum + Sign * x ^ k / k
    Sign = -1 * Sign
NEXT k
PRINT N, Sum
```

N	$P_N(.5)$
1	.500000000
5	.464583333
9	.463684275
13	.463649276
17	.463647690
19	.463647590
21	.463647613
23	.463647608
25	.463647609
27	.463647609

Thus, $\arctan(.5) = .4636476$ to seven decimal places, and we needed the Taylor polynomial of degree 21 to get this level of accuracy.

4. b. When $x = 2$, the terms in the Taylor series blow up. The graphs of $\arctan(x)$, $P_5(x)$, and $P_7(x)$ suggest that this series converges only for $-1 \leq x \leq 1$.



To get around this difficulty, let $\theta = \arctan(x)$. Then $x = \tan(\theta) = 1/\cot(\theta) = 1/\tan(\frac{\pi}{2} - \theta)$, so $\tan(\frac{\pi}{2} - \theta) = 1/x$. This implies $\frac{\pi}{2} - \theta = \arctan(1/x)$ and, finally, that $\arctan(x) = \frac{\pi}{2} - \arctan(1/x)$.

Thus $\arctan(2) = \frac{\pi}{2} - \arctan(.5)$. We just calculated an estimate for $\arctan(.5)$ using Taylor series. We can use it to get a value for $\arctan(2)$:

$$\begin{aligned}\arctan(2) &= \frac{\pi}{2} - \arctan(.5) = 1.5707963 - .4636476 \\ &= 1.1071487\end{aligned}$$

Since the last digit is rounded in each case, we can only be sure of six decimal places; in fact, though, all seven decimal places are correct.

5. a. We are asked to calculate

$$4 \cdot \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right),$$

to three decimal places accuracy. We can do this using the program in the solution to exercise 4.

N	$4 \cdot P_N(1)$
1001	3.143588
2001	3.142592
3001	3.142259
4001	3.142092
4501	3.142036
4901	3.142001
4903	3.141185
4905	3.142000318
4907	3.141185156
4909	3.141999996
4911	3.141184875
4913	3.141999654

Thus $\pi = 3.141 \dots$. Since terms are alternately too large and too small, the table makes it clear that we must go out at least to $N = 4909$ to be certain that the value of the third decimal is 1.

5. b. Let us define α , β , and γ by

$$\begin{aligned}\alpha &= \arctan(1/2) & \tan(\alpha) &= 1/2 \\ \beta &= \arctan(1/5) & \tan(\beta) &= 1/5 \\ \gamma &= \arctan(1/8) & \tan(\gamma) &= 1/8\end{aligned}$$

Our aim is to show $\alpha + \beta + \gamma = \pi/4$. We will do this by showing $\tan(\alpha + \beta + \gamma) = 1 = \tan(\pi/4)$. First consider $\alpha + \beta$. Using the given formula for the tangent of the sum of two angles, we get

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)} \\ &= \frac{1/2 + 1/5}{1 - 1/10} = \frac{7/10}{9/10} = 7/9.\end{aligned}$$

Using the formula a second time with the angles $\alpha + \beta$ and γ , we get

$$\begin{aligned}\tan(\alpha + \beta + \gamma) &= \frac{\tan(\alpha + \beta) + \tan(\gamma)}{1 - \tan(\alpha + \beta)\tan(\gamma)} \\ &= \frac{7/9 + 1/8}{1 - 7/72} = \frac{65/72}{65/72} = 1.\end{aligned}$$

Since $0 < \alpha + \beta + \gamma < \pi/2$ and $\tan(\alpha + \beta + \gamma) = \tan(\pi/4)$, it follows that $\alpha + \beta + \gamma = \pi/4$.

Here are estimates for $\arctan(1/2)$, $\arctan(1/5)$, and $\arctan(1/8)$ that stabilize to twelve decimal places:

N	$P_N(1/2)$	$P_N(1/5)$
1	1.333333333333	.200000000000
3	.458333333333	.197333333333
5	.464583333333	.197397333333
7	.463467261904	.197395504761
9	.463684275793	.197395561650
11	.463639886589	.197395559788
13	.463649276613	.197395559852
15	.463647242107	.197395559849
17	.463647690895	.197395559849
19	.463647590509	.197395559849
21	.463647613215	
23	.463647608032	
25	.463647609224	
27	.463647608948	
29	.463647609012	
31	.463647608997	
33	.463647609001	
35	.463647609000	
37	.463647609000	
39	.463647609000	

N	$P_N(1/8)$
1	.125000000000
3	.124348958333
5	.124355061848
7	.124354993729
9	.124354994557
11	.124354994546
13	.124354994546
15	.124354994546

To construct our estimate for π , we round each of these numbers to 13 decimal places, add, and multiply by 4:

$$\begin{aligned}&.4636476090008 \\ &.1973955598499 \\ &+.1243549945468 \\ &\hline &.7853981633967 \times 4 = 3.1415926535900\end{aligned}$$

Therefore, we estimate that $\pi = 3.141592653590$ to 12 decimal places. In fact,

$$\pi = \underline{3.14159265358979} \dots$$

When this is rounded to twelve decimal places, it gives the value we calculated.

The tables show how far we had to go to estimate each of the three numbers. The estimate for $\arctan(1/2)$ took the longest to converge because $1/2$ is the largest of the three numbers, so its powers (which appear in the series for the arctan function) took the longest to become small.

6. a. If e^x is to continue to have a power series when x is an imaginary number is , then

$$\begin{aligned}e^{is} &= 1 + is + \frac{i^2 s^2}{2!} + \frac{i^3 s^3}{3!} + \frac{i^4 s^4}{4!} \\ &\quad + \frac{i^5 s^5}{5!} + \frac{i^6 s^6}{6!} + \frac{i^7 s^7}{7!} + \dots \\ &= \left(1 - \frac{s^2}{2!} + \frac{s^4}{4!} - \frac{s^6}{6!} + \dots\right) \\ &\quad + i \left(s - \frac{s^3}{3!} + \frac{s^5}{5!} - \frac{s^7}{7!} + \dots\right) \\ &= \cos(s) + i \sin(s)\end{aligned}$$

To finish, we needed to recognize the Taylor series for $\cos(s)$ and $\sin(s)$.

$$\mathbf{6. b.} \quad e^{\pi i} = \cos(\pi) + i \sin(\pi) = -1 + i \cdot 0 = -1.$$

6. c. According to the law of exponents that says $x^{a+b} = x^a \cdot x^b$, we have $e^{r+si} = e^r \cdot e^{si} = e^r (\cos(s) + i \sin(s))$.

6. d. Since $-5 = 5 \cdot -1 = 5 \cdot e^{\pi i}$, we just need to choose r so $e^r = 5$. Thus $r = \ln(5)$ and $e^{\ln 5 + \pi i} = e^{\ln 5} \cdot e^{\pi i} = 5 \cdot -1 = -5$: $r + si = \ln 5 + \pi i$.

7. a. Get the series for $\cosh(x)$ by adding the series for e^x and e^{-x} (and dividing by 2):

$$\begin{aligned}e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots \\ e^{-x} &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots \\ \cosh(x) &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\end{aligned}$$

Apart from sign, these are exactly the same terms as appear in the Taylor series for $\cos(x)$ centered at $x = 0$. The terms in $\cos(x)$ alternate in sign while these don't.

7. b. Get the series for $\sinh(x)$ by subtracting the series for e^{-x} from the series for e^x (and dividing by 2):

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \\ e^{-x} &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots \\ \sinh(x) &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \end{aligned}$$

Apart from sign, these are exactly the same terms as appear in the Taylor series for $\sin(x)$ centered at $x = 0$. The terms in $\sin(x)$ alternate in sign while these don't.

7. c. Let $x = \cosh(t)$, $y = \sinh(t)$. Then

$$\begin{aligned} x^2 &= \left(\frac{e^t + e^{-t}}{2} \right)^2 = \frac{e^{2t} + 2 + e^{-2t}}{4} \\ y^2 &= \left(\frac{e^t - e^{-t}}{2} \right)^2 = \frac{e^{2t} - 2 + e^{-2t}}{4} \\ x^2 - y^2 &= \frac{2 - (-2)}{4} = 1 \end{aligned}$$

Thus, the point $(x, y) = (\cosh(t), \sinh(t))$ lies on the hyperbola $x^2 - y^2 = 1$.

8. The series for $(1+x)^c$ can be inferred from the Taylor polynomials constructed in exercise 11 of 10.2:

$$(1+x)^c = 1 + cx + \frac{c(c-1)}{2!}x^2 + \frac{c(c-1)(c-2)}{3!}x^3 + \cdots$$

8. a. Notice that every coefficient from x^3 onward contains the factor $(c-2)$. Therefore, if $c = 2$, that factor is zero, and so the whole term is zero. This means that the infinite series reduces to a finite sum of three terms:

$$(1+x)^2 = 1 + 2x + \frac{2 \cdot 1}{2!}x^2 = 1 + 2x + x^2.$$

This is just the ordinary binomial expansion; it is a reasonable result.

8. b. Every coefficient from x^4 onward has the factor $(c-3)$, and this is 0 if $c = 3$. Therefore the series collapses to a finite sum again, and

$$(1+x)^3 = 1 + 3x + \frac{3 \cdot 2}{2!}x^2 + \frac{3 \cdot 2 \cdot 1}{3!}x^3 = 1 + 3x + 3x^2 + x^3.$$

Once again, this is just what we would want to be true.

8. c. Every coefficient from x^{n+1} onward has the factor $(c-n)$, and this is 0 if $c = n$ (assuming n is a positive integer). Therefore the series collapses to a finite sum that has the following form:

$$\begin{aligned} (1+x)^n &= 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \\ &\quad + \frac{n(n-1)\cdots 4 \cdot 3}{(n-2)\cdots 3 \cdot 2}x^{n-2} \\ &\quad + \frac{n(n-1)\cdots 3 \cdot 2}{(n-1)!}x^{n-1} + x^n \\ &= 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots \\ &\quad + \frac{n(n-1)}{2}x^{n-2} + nx^{n-1} + x^n \end{aligned}$$

8. d. Since $1/(1-u) = (1-u)^{-1}$, we can use the series for $(1+x)^c$ if we let $c = -1$ and $x = -u$. Then

$$\begin{aligned} \frac{1}{1-u} &= (1+(-u))^{-1} \\ &= 1 + (-1)(-u) + \frac{(-1)(-2)}{2!}(-u)^2 \\ &\quad + \frac{(-1)(-2)(-3)}{3!}(-u)^3 + \cdots \\ &= 1 + u + u^2 + u^3 + \cdots \end{aligned}$$

This agrees with the series for $1/(1-u)$ that was given in the text (page 622) and determined there by other means.

8. e. Let $x = 16$ and $c = 1/2$. We find that successive terms in the series for $(1+16)^{1/2}$ grow without bound, and the partial sums of the series fail to converge. The sum of the series is infinite,

8. f. If we let $x = 1/16$ and $c = 1/2$, then

$$\begin{aligned} (1 + \frac{1}{16})^{1/2} &= 1 + \frac{1}{2} \frac{1}{16} + \frac{(\frac{1}{2})(-\frac{1}{2})}{2!} \frac{1}{16^2} \\ &\quad + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!} \frac{1}{16^3} + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!} \frac{1}{16^4} + \cdots \\ &= 1 + \frac{1}{2 \cdot 16} - \frac{1}{2^2 \cdot 2! \cdot 16^2} + \frac{1 \cdot 3}{2^3 \cdot 3! \cdot 16^3} \\ &\quad - \frac{1 \cdot 3 \cdot 5}{2^4 \cdot 4! \cdot 16^4} + \cdots \end{aligned}$$

The following program calculates this sum; note how the factors 1, 3, 5..., that appear in the numerators are handled in the program.

```

DEFDBL A-Z
N = 4
x = 1 / 16
Sum = 1 + 1 / 32
Sign = -1
a = 1
b = 1
DEF fnfact(m)
  p = 1
  FOR r = 2 TO m
    p = p * r
  NEXT r
  fnfact = p
END DEF
FOR k = 2 TO N
  a = a * b
  denom = 2 ^ k * fnfact(k) * 16 ^ k
  Sum = Sum + Sign * a / denom
  b = b + 2
  Sign = -1 * Sign
NEXT k
PRINT N, 4 * Sum

```

N	$P_N(1/16)$	$4P_N(1/16)$
1	1.031250000	4.125000000
2	1.030761719	4.123046875
3	1.030776978	4.123107910
4	1.030776381	4.123105526
5	1.030776408	4.123105630
6	1.030776406	4.123105625
7	1.030776406	4.123105626

We find that $\sqrt[3]{17} = 4.1231056$ to seven decimal places accuracy, and we obtain that accuracy with a Taylor polynomial of degree 6.

8. g. We can write

$$\begin{aligned}\sqrt[3]{30} &= \sqrt[3]{27+3} = \sqrt[3]{27\left(1+\frac{1}{9}\right)} = \sqrt[3]{27} \cdot \sqrt[3]{1+\frac{1}{9}} \\ &= 3 \cdot \sqrt[3]{1+\frac{1}{9}} = 3 \cdot \left(1+\frac{1}{9}\right)^{\frac{1}{3}}.\end{aligned}$$

This suggests we should write out the series for the function $(1+x)^c$ with $x = 1/9$ and $c = 1/3$:

$$\begin{aligned}\left(1+\frac{1}{9}\right)^{1/3} &= 1 + \frac{1}{3} \cdot \frac{1}{9} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!} \cdot \frac{1}{9^2} \\ &\quad + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!} \cdot \frac{1}{9^3} + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{4!} \cdot \frac{1}{9^4} + \cdots \\ &= 1 + \frac{1}{3 \cdot 9} - \frac{2}{3^2 \cdot 2! \cdot 9^2} + \frac{2 \cdot 5}{3^3 \cdot 3! \cdot 9^3} \\ &\quad - \frac{2 \cdot 5 \cdot 8}{3^4 \cdot 4! \cdot 9^4} + \cdots\end{aligned}$$

By modifying the program in part f we can obtain the following table of estimates.

N	$3 \cdot P_N(1/9)$
1	3.111111111
2	3.106995885
3	3.107249911
4	3.107231094
5	3.107232628
6	3.107232495
7	3.107232507
8	3.107232506

Thus $\sqrt[3]{30} = 3.1072325$ to seven decimal places, and this was determined using a Taylor polynomial of degree 8.

Evaluating Taylor Series Rapidly

9. If we write $n = 2k + 1$, then we can summarize the discussion that precedes exercise 9 in the following way:

n	k	operations needed
11	5	$4(1+2+3+4+5) = 60$
51	25	$4(1+2+\cdots+25) = 1300$

Therefore, for any $n = 2k + 1$, we see that the number of operations is

$$4(1+2+\cdots+k) = 4 \frac{k(k+1)}{2} = \frac{4k^2+4k}{2} = \frac{n^2-1}{2}.$$

(We used the fact that $n^2 = (2k+1)^2 = 4k^2 + 4k + 1$.) When n is large, $(n^2-1)/2$ is essentially the same as $n^2/2$; this completes the proof.

10. Work from the inside out, one set of parentheses at a time. To get the value of the expression in the innermost parentheses, we must multiply x by x , then divide by 10, then divide by 11. This takes three operations altogether.

Now go out one level. It also takes three operations to get $x^2/8 \cdot 9$, but then it takes one more to multiply this by the contents of the previous (inner) parenthetical expression. This takes four operations altogether.

$$\underbrace{1 - \frac{x^2}{8 \cdot 9} \left(\underbrace{1 - \frac{x^2}{10 \cdot 11}}_{3 \text{ operations}} \right)}_{4 \text{ more operations}}$$

At each successive level we must do the same thing, using four additional operations in the process. Finally, we need one more operation to multiply x by the value of the outermost parentheses. The total is therefore

$$3 + 4 + 4 + 4 + 4 + 1 = 4 \times 5 = 20.$$

Note that there were 5 sets of parentheses, and $11 = 2 \cdot 5 + 1$. Note also that we count each computation of x^2 as a separate operation. If, instead, we store the value of x^2 after the first computation, then the successive stages involve only three operations each, not four. This produces a total of $3 \times 5 + 1 = 16$. This point is pursued further in the answer to exercise 12.

11. If we evaluate $P_{51}(x)$ using the same strategy, we need 25 pairs of nested parentheses. Since the first parenthetical expression requires three operations, every subsequent one requires four, and there is a final multiplication, the total number of operations is

$$3 + \underbrace{4 + \cdots + 4}_{24 \text{ times}} + 1 = 4 \times 25 = 100.$$

Since the standard way of evaluating $P_{51}(x)$ uses 1300 operations, the new strategy is about 13 times faster.

12. Using the new strategy, we can write the polynomial $P_n(x)$ using k pairs of nested parentheses, where $n = 2k + 1$. The number of operations is then $4k = 4(n - 1)/2 = 2(n - 1) \approx 2n$.

The new strategy for evaluating $P_n(x)$ is therefore about

$$\frac{n^2/2}{2n} = \frac{n}{4}$$

times as fast as the standard one. (If $n = 11$, then $n/4 \approx 3$; if $n = 51$, then $n/4 \approx 13$. These results agree with the calculations already done.)

Note: Had we stored the value of x^2 , instead of recalculating it each time, the new strategy would take $3k + 1 \approx 3n/2$ operations, and the this approach would have been about $(n^2/2)/(3n/2) = n/3$ times as fast as the standard one. In each case, the factor of speed increase would have been proportional to n .

13.a. The following QuickBasic program, which is a simplification of the program TAYLOR on page 600 of the text, is designed to calculate e^1 from the Taylor series for e^x using the standard approach.

```
DEFDBL A-Z
N = 300
x = 1
sum = 1
DEF fnfact(m)
  p = 1
  FOR r = 2 TO m
    p = p * r
  NEXT r
  fnfact = p
END DEF
FOR k = 1 TO N
  sum = sum + x ^ k / fnfact(k)
NEXT k
PRINT N, sum
```

In fact, QuickBasic on my machine (a 486) cannot manage this calculation (because $k!$ gets too large: $300! = 3.0606 \times 10^{614}$). The largest value of N it permits is about 170. When $N = 170$, the calculation takes about one second and gives the value 2.718281828459046.

The new approach starts from

$$P_{300}(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{299!} + \frac{1}{300!} \\ = 1 + 1 + \frac{1}{2} \left(1 + \frac{1}{3} \left(1 + \cdots \left(1 + \frac{1}{300} \right) \cdots \right) \right).$$

The program is much simpler:

```
DEFDBL A-Z
N = 300
x = 1
paren = 1
FOR k = N TO 1 STEP - 1
  paren = 1 + paren * x / k
NEXT k
PRINT N, paren
```

This time my machine has no trouble dealing with $N = 300$, and it makes the calculation without any obvious pause. It gives the value 2.718281828459045.

13.b. The following QuickBasic program will plot $y = e^x$ from the 170-th degree Taylor polynomial using *both* approaches. The standard approach (which plots `fnslow`) takes about 156 seconds in QuickBasic on a 486 machine, while the new approach (which plots `fnquik`) takes only about 3 seconds. Times will vary on other machines, but the speed-up factor should be similar to what it is here—namely, about 50. Note that $N = 170$ here because my computer couldn't handle a larger value.

```
SCREEN 9
COLOR 8, 7
WINDOW (0, 0)-(2, 8)
DEFDBL A-Z
N = 170
DEF fnfact(m)
  p = 1
  FOR r = 2 TO m
    p = p * r
  NEXT r
  fnfact = p
END DEF
DEF fnslow(x)
  sum = 1
  FOR k = 1 TO N
    sum = sum + x ^ k / fnfact(k)
  NEXT k
  fnslow = sum
END DEF
DEF fnquik(x)
```

```

p = 1
FOR k = N TO 1 STEP -1
  p = 1 + p * x / k
NEXT k
fnquik = p
END DEF
FOR x = 0 TO 2 STEP .01
  LINE (x, fnquik(x))-(x + .01,
        fnquik(x + .01)), 12
NEXT x
FOR x = 0 TO 2 STEP .01
  LINE (x, fnslow(x))-(x + .01,
        fnslow(x + .01)), 14
NEXT x

```

In this program, the background is color 7 and the default foreground is color 8; `fnquik` is drawn in color 12 and `fnslow` in color 14.

10.4 Power Series and Differential Equations

1. a. We follow the method of the text, using $y = a_0 + a_1x + a_2x^2 + \dots$.

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \dots$$

$$2xy = 2a_0x + 2a_1x^2 + 2a_2x^3 + 2a_3x^4 + \dots$$

Since two power series are equal only if the coefficients of corresponding powers of x are equal, we find

$$\begin{array}{ll}
 a_1 = 0 & a_1 = 0 \\
 2a_2 = 2a_0 & a_2 = a_0 \\
 3a_3 = 2a_1 & a_3 = 0 \\
 4a_4 = 2a_2 & a_4 = \frac{1}{2}a_2 = \frac{1}{2}a_0 \\
 5a_5 = 2a_3 & a_5 = 0 \\
 6a_6 = 2a_4 & a_6 = \frac{1}{3}a_4 = \frac{1}{3!}a_0 \\
 7a_7 = 2a_5 & a_7 = 0 \\
 8a_8 = 2a_6 & a_8 = \frac{1}{4}a_6 = \frac{1}{4!}a_0
 \end{array}$$

Thus

$$y = a_0 \left(1 + x^2 + \frac{1}{2!}x^4 + \frac{1}{3!}x^6 + \dots + \frac{1}{n!}x^{2n} + \dots \right).$$

(In fact, the formula makes it clear that $y = a_0e^{x^2}$, and we can independently check that this function satisfies the original differential equation.)

1. b. Again, take $y = a_0 + a_1x + a_2x^2 + \dots$.

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \dots$$

$$3x^2y = 3a_0x^2 + 3a_1x^3 + 3a_2x^4 + \dots$$

$$\begin{array}{ll}
 a_1 = 0 & a_1 = 0 \\
 2a_2 = 0 & a_2 = 0 \\
 3a_3 = 3a_0 & a_3 = a_0 \\
 4a_4 = 3a_1 & a_4 = 0 \\
 5a_5 = 3a_2 & a_5 = 0 \\
 6a_6 = 3a_3 & a_6 = \frac{1}{2}a_3 = \frac{1}{2}a_0 \\
 7a_7 = 3a_4 & a_7 = 0 \\
 8a_8 = 3a_5 & a_8 = 0 \\
 9a_9 = 3a_6 & a_9 = \frac{1}{3}a_6 = \frac{1}{3!}a_0
 \end{array}$$

Thus

$$y = a_0 \left(1 + x^3 + \frac{1}{2!}x^6 + \frac{1}{3!}x^9 + \dots + \frac{1}{n!}x^{3n} + \dots \right),$$

and this is $a_0e^{x^3}$.

1. c. This time, we must add the coefficients of like powers and set each sum equal to zero:

$$\begin{array}{rcl}
 y'' = & 2a_2 + & 3 \cdot 2a_3x + & 4 \cdot 3a_4x^2 + & 5 \cdot 4a_5x^3 + & \dots \\
 xy = & & a_0x + & a_1x^2 + & a_2x^3 + & \dots
 \end{array}$$

Note that the sum of the coefficients of x^n looks like this: $(n+1)(n+2)a_{n+2} + a_{n-1}$.

$$\begin{array}{l}
 2a_2 = 0 \\
 2 \cdot 3a_3 + a_0 = 0 \\
 3 \cdot 4a_4 + a_1 = 0 \\
 4 \cdot 5a_5 + a_2 = 0 \\
 5 \cdot 6a_6 + a_3 = 0 \\
 6 \cdot 7a_7 + a_4 = 0 \\
 7 \cdot 8a_8 + a_5 = 0
 \end{array}$$

In this recursion, a_n is related back to a_{n-3} , so it is helpful to write the relations in three separate columns, as follows:

$$\begin{array}{lll}
 a_2 = 0 & a_3 = -\frac{1}{2 \cdot 3}a_0 & a_4 = -\frac{1}{3 \cdot 4}a_1 \\
 a_5 = 0 & a_6 = \frac{1}{2 \cdot 3 \cdot 5 \cdot 6}a_0 & a_7 = \frac{1}{3 \cdot 4 \cdot 6 \cdot 7}a_1 \\
 a_8 = 0 & a_9 = \dots &
 \end{array}$$

(Note that we skipped over intermediate steps of the form $a_6 = -a_3/(5 \cdot 6) = +a_0/(2 \cdot 3 \cdot 5 \cdot 6)$.)

This work shows that y is the sum of two separate series: one has a_0 as a constant multiplier and involves powers of x that are divisible by 3; the other has a_1 as constant multiplier and involves powers of x that leave a remainder of +1 when divided by 3:

$$y = a_0 \left(1 - \frac{1}{2 \cdot 3} x^3 + \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} x^6 + \dots \right) + a_1 \left(x - \frac{1}{3 \cdot 4} x^4 + \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 + \dots \right)$$

The denominators look something like factorials; we can turn them into factorials by including the missing numbers in the denominators and balancing the expression by putting them in the numerator as well. In this way we can write the general term for each series:

$$a_0 \cdot (-1)^n \frac{1 \cdot 4 \cdot 7 \cdots (3n-2)}{(3n)!} x^{3n} \\ a_1 \cdot (-1)^n \frac{2 \cdot 5 \cdot 8 \cdots (3n-1)}{(3n+1)!} x^{3n+1}.$$

1. d. Again, add coefficients of like powers, and set each sum equal to zero:

$$\begin{array}{rclcl} y & = & a_0 + & a_1 x + & a_2 x^2 + & a_3 x^3 + & \dots \\ xy' & = & & a_1 x + & 2a_2 x^2 + & 3a_3 x^3 + & \dots \\ y'' & = & 2a_2 + & 2 \cdot 3a_3 x + & 3 \cdot 4a_4 x^2 + & 4 \cdot 5a_5 x^3 + & \dots \end{array}$$

We find

$$\begin{array}{ll} 2a_2 + a_0 = 0 & a_2 = -\frac{1}{2}a_0 \\ 2a_1 + 2 \cdot 3a_3 = 0 & a_3 = -\frac{1}{3}a_1 \\ 3a_2 + 3 \cdot 4a_4 = 0 & a_4 = -\frac{1}{4}a_2 = \frac{1}{2 \cdot 4}a_0 \\ 4a_3 + 4 \cdot 5a_5 = 0 & a_5 = -\frac{1}{5}a_3 = \frac{1}{3 \cdot 5}a_1 \\ 5a_4 + 5 \cdot 6a_6 = 0 & a_6 = -\frac{1}{6}a_4 = -\frac{1}{2 \cdot 4 \cdot 6}a_0 \end{array}$$

Once again y splits into two series: one has only even powers and has a_0 as constant multiplier; the other has only odd powers and has a_1 as constant multiplier.

$$y = a_0 \left(1 - \frac{1}{2} x^2 + \frac{1}{2 \cdot 4} x^4 - \frac{1}{2 \cdot 4 \cdot 6} x^6 + \dots \right) + a_1 \left(x - \frac{1}{3} x^3 + \frac{1}{3 \cdot 5} x^5 - \frac{1}{1 \cdot 3 \cdot 5 \cdot 7} x^7 + \dots \right)$$

We can get relatively simple expressions for the general term in each case. First, factor out a 2 from each factor in the coefficient of x^{2n} to get

$$\begin{aligned} a_0 \cdot (-1)^n \frac{1}{2 \cdot 4 \cdots 2n} x^{2n} \\ = a_0 \cdot (-1)^n \frac{1}{2 \cdot 1 \cdot 2 \cdot 2 \cdots 2 \cdot n} x^{2n} \\ = a_0 \cdot (-1)^n \frac{1}{2^n n!} x^{2n} \end{aligned}$$

Then, by filling in factors in the numerator and denominator of the coefficient of x^{2n+1} and using the result we just got, we find

$$\begin{aligned} a_1 \cdot (-1)^n \frac{1}{1 \cdot 3 \cdots (2n+1)} x^{2n+1} \\ = a_1 \cdot (-1)^n \frac{2 \cdot 4 \cdots 2n}{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n+1)} x^{2n+1} \\ = a_1 \cdot (-1)^n \frac{2^n n!}{(2n+1)!} x^{2n+1} \end{aligned}$$

2. a. We set the following coefficients equal to each other, term by term:

$$\begin{array}{rclcl} y'' & = & 2a_2 + & 2 \cdot 3a_3 x + & 3 \cdot 4a_4 x^2 + & 4 \cdot 5a_5 x^3 + & \dots \\ -y & = & -a_0 - & a_1 x - & a_2 x^2 - & a_3 x^3 - & \dots \end{array}$$

Thus

$$\begin{array}{ll} 2a_2 = -a_0 & a_2 = -\frac{1}{2}a_0 \\ 2 \cdot 3a_3 = -a_1 & a_3 = -\frac{1}{3!}a_1 \\ 3 \cdot 4a_4 = -a_2 & a_4 = \frac{1}{4!}a_0 \\ 4 \cdot 5a_5 = -a_3 & a_5 = \frac{1}{5!}a_1 \\ 5 \cdot 6a_6 = -a_4 & a_6 = -\frac{1}{6!}a_0 \\ 6 \cdot 7a_7 = -a_5 & a_7 = -\frac{1}{7!}a_1 \end{array}$$

and the solution to the differential equation is

$$y = a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right) + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right)$$

2. b. The series of even powers is the Taylor series for $\cos(x)$, and the series of odd powers is the Taylor series for $\sin(x)$, both centered at $x = 0$.

3. a. The work here is virtually identical to the work in the previous exercise. However, since the differential equation is $y'' = +y$ instead of $y'' = -y$, no minus signs appear in the recursion relations. Therefore the solution is

$$y = a_0 \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \right) + a_1 \left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \right)$$

3. b. The series of even powers is the Taylor series for $\cosh(x)$, and the series of odd powers is the Taylor series for $\sinh(x)$, both centered at $x = 0$.

4. a. This is virtually the same as exercise 1a. However, since the differential equation is $y' = xy$ instead of $y' = 2xy$, the recursion relations are missing a factor of 2 on the right hand side:

$$\begin{aligned} a_1 &= 0 & a_1 &= 0 \\ 2a_2 &= a_0 & a_2 &= \frac{1}{2}a_0 \\ 3a_3 &= a_1 & a_3 &= 0 \\ 4a_4 &= a_2 & a_4 &= \frac{1}{4 \cdot 2}a_0 = \frac{1}{2! \cdot 2^2}a_0 \\ 5a_5 &= a_3 & a_5 &= 0 \\ 6a_6 &= a_4 & a_6 &= \frac{1}{6 \cdot 4 \cdot 2}a_0 = \frac{1}{3! \cdot 2^3}a_0 \end{aligned}$$

We have used the trick from exercise 1d to reduce $2 \cdot 4 \cdot 2n$ to $n! \cdot 2^n$. The solution is

$$\begin{aligned} y &= a_0 \left(1 + \frac{1}{2}x^2 + \frac{1}{2! \cdot 2^2}x^4 + \frac{1}{3! \cdot 2^3}x^6 + \cdots \right) \\ &= a_0 \left(1 + \frac{x^2}{2} + \frac{1}{2!} \left(\frac{x^2}{2} \right)^2 + \frac{1}{3!} \left(\frac{x^2}{2} \right)^3 + \cdots \right) \end{aligned}$$

4. b. If $y = e^{x^2/2}$ then the chain rule tells us $y' = (x^2/2)' \cdot e^{x^2/2} = x \cdot e^{x^2/2} = xy$. Therefore, $y = e^{x^2/2}$ does indeed satisfy the differential equation $y' = xy$. Furthermore, the Taylor series for $e^{x^2/2}$, centered at $x = 0$, can be obtained from the Taylor series for e^u by substituting $u = x^2/2$:

$$\begin{aligned} e^u &= 1 + u + \frac{1}{2!}u^2 + \frac{1}{3!}u^3 + \cdots \\ e^{x^2/2} &= 1 + \frac{x^2}{2} + \frac{1}{2!} \left(\frac{x^2}{2} \right)^2 + \frac{1}{3!} \left(\frac{x^2}{2} \right)^3 + \cdots \end{aligned}$$

This is exactly the series contained in parentheses that was obtained in the first part of this exercise. Thus, the solution found there is $y = a_0 e^{x^2/2}$.

5. a. When $p = 1$, the Bessel equation takes the form $x^2 \cdot y'' + x \cdot y' + (x^2 - 1) \cdot y = 0$. To find the solution we assume $y = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$. The various terms begin in the following way:

$$\begin{array}{rccccccc} x^2 y'' & = & & + & 2a_2 x^2 & + & 2 \cdot 3a_3 x^3 & + & 3 \cdot 4a_4 x^4 \\ x y' & = & & + & a_1 x & + & 2a_2 x^2 & + & 3a_3 x^3 & + & 4a_4 x^4 \\ x^2 y & = & & + & a_0 x^2 & + & a_1 x^3 & + & a_2 x^4 & + & a_3 x^5 \\ -y & = & -a_0 & - & a_1 x & - & a_2 x^2 & + & -a_3 x^3 & + & -a_4 x^4 \end{array}$$

In each column, the sum must be zero. Therefore,

$$\begin{aligned} a_0 &= 0 & a_0 &= 0 \\ 3a_2 + a_0 &= 0 & a_2 &= 0 \\ 2 \cdot 4a_3 + a_1 &= 0 & a_3 &= -\frac{1}{2 \cdot 4}a_1 \\ 3 \cdot 5a_4 + a_2 &= 0 & a_4 &= 0 \\ 4 \cdot 6a_5 + a_3 &= 0 & a_5 &= \frac{1}{2 \cdot 4^2 \cdot 6}a_1 \\ 5 \cdot 7a_6 + a_4 &= 0 & a_6 &= 0 \\ 6 \cdot 8a_7 + a_5 &= 0 & a_7 &= -\frac{1}{2 \cdot 4^2 \cdot 6^2 \cdot 8}a_1 \end{aligned}$$

All even powers have coefficient 0, and $y =$

$$a_1 \left(x - \frac{1}{2 \cdot 4}x^3 + \frac{1}{2 \cdot 4^2 \cdot 6}x^5 - \frac{1}{2 \cdot 4^2 \cdot 6^2 \cdot 8}x^7 + \cdots \right)$$

Since $y' = a_1 + O(x^2)$, it follows that $y' = a_1$ when $x = 0$. Therefore, to construct the Bessel function $J_1(x)$ using the property $J_1'(0) = 1/2$, we should take $a_1 = 1/2$ in the series solution:

$$\begin{aligned} J_1(x) &= \\ \frac{1}{2}x - \frac{1}{2^2 \cdot 4}x^3 + \frac{1}{2^2 \cdot 4^2 \cdot 6}x^5 - \frac{1}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8}x^7 + \cdots \end{aligned}$$

The pattern shows us what to expect for the coefficient of x^{2n+1} :

$$(-1)^n \frac{1}{2^2 \cdot 4^2 \cdot 6^2 \cdots (2n)^2 \cdot (2n+2)}$$

In fact, we can rewrite this as

$$\begin{aligned} &(-1)^n \frac{1}{2 \cdot 4 \cdot 6 \cdots (2n)} \cdot \frac{1}{2 \cdot 4 \cdot 6 \cdots (2n) \cdot (2n+2)} \\ &= (-1)^n \frac{1}{2^n n!} \cdot \frac{1}{2^{n+1} (n+1)!} = (-1)^n \frac{1}{2^{2n+1} n! (n+1)!} \end{aligned}$$

From this point of view, the series for $J_1(x)$ takes the form

$$\begin{aligned} J_1(x) &= \frac{x}{2} - \frac{x^3}{2^3 1! 2!} + \frac{x^5}{2^5 2! 3!} - \frac{x^7}{2^7 3! 4!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2^{2n+1} n! (n+1)!} \end{aligned}$$

which appears in reference manuals as one of the standard forms for $J_1(x)$.

5. b. We'll show that the derivative of the term of degree $2n+2$ in $J_0(x)$ is the same as the negative of the term of degree $2n+1$ in $J_1(x)$. From the expression for $J_0(x)$ on page 634 in the text, we see that the term of degree $2n+2$ is

$$(-1)^{n+1} \frac{x^{2n+2}}{2^{2n+2}((n+1)!)^2}.$$

Its derivative is

$$\begin{aligned} & (-1)^{n+1} \frac{(2n+2)x^{2n+1}}{2^{2n+2}((n+1)!)^2} = \\ & = (-1)^{n+1} \frac{2(n+1)x^{2n+1}}{2 \cdot 2^{2n+1}(n+1)!(n+1)!} \\ & = -1 \cdot (-1)^n \frac{x^{2n+1}}{2^{2n+1}n!(n+1)!} \\ & = -\{\text{term of degree } 2n+1 \text{ of } J_1(x)\} \end{aligned}$$

5. c. Differentiate the given equation using the product rule:

$$x \cdot J_0''' + J_0'' + J_0'' + x \cdot J_0' + J_0 = 0.$$

(Don't add together the two appearances of J_0'' .) Now multiply by x :

$$x^2 \cdot J_0''' + x \cdot J_0'' + x \cdot J_0'' + x^2 \cdot J_0' + x \cdot J_0 = 0.$$

Since we can rewrite the original equation in the form $xJ_0'' + xJ_0 = -J_0'$, we see that we can replace the underlined terms by the single term $-J_0'$:

$$x^2 \cdot J_0''' + x \cdot J_0'' + x^2 \cdot J_0' - J_0' = 0.$$

This is the same as

$$x^2 \cdot (J_0')'' + x \cdot (J_0')' + (x^2 - 1)J_0' = 0,$$

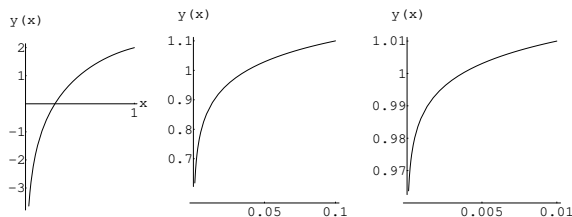
and it shows that $J_0'(x)$, and hence $-J_0'(x)$, is a solution to the Bessel equation with $p = 1$. Finally, $-J_0'(0) = -\frac{1}{2}$, so $-J_0' = J_1$.

6. a. The 0-order Bessel equation is $x \cdot y'' + y' + x \cdot y = 0$. However, when $x = 0$ this becomes simply $y' = 0$. Therefore, every solution $y(x)$ must have $y'(0) = 0$; in particular, $y'(0) = 1$ is impossible.

6. b. To use Euler's method, we must first rewrite the differential equation as a pair of first-order differential equations. Let $y' = v$; then we have $xv' + v + xy = 0$, so the equations are

$$\begin{aligned} y' &= v \\ v' &= -y - \frac{v}{x}. \end{aligned}$$

This implies $v' \rightarrow \pm\infty$ as $x \rightarrow 0$. In fact, the graphs of the solutions become vertical as $x \rightarrow 0$, and $y(x)$ is undefined for $x < 0$:



These figures were plotted using Mathematica's numerical differential equation solver (which uses Runge-Kutta methods rather than the simpler Euler's method). Here are some tabulations using Euler's method.

$$y(1) = 2, \quad y'(1) = 1, \quad \Delta x = -.1$$

x	y	v	x	y	v
1.0	2.00	1.00	.4	.86	3.73
.9	1.90	1.30	.3	.48	4.75
.8	1.77	1.63	.2	.01	6.39
.7	1.61	2.02	.1	-.63	9.58
.6	1.40	2.46	0.0	-1.59	19.1
.5	1.16	3.02			

Note that we find $y(0) = -1.59$ only because we have taken $\Delta x = -.1$. In the following table we take smaller step sizes and find that Euler estimates for $y(0)$ decrease toward $-\infty$.

p	$y(0)$
5	-3.2
10	-8.6
15	-14.3
20	-20.0

For parts ii and iii, we compute only the Euler estimates for $y(0)$ for various step sizes.

ii. $\Delta x = -.1 \times 2^{-p}$		iii. $\Delta x = -.01 \times 2^{-p}$	
p	$y(0)$	p	$y(0)$
5	.767	5	.9780
10	.417	10	.9445
15	.053	15	.9098
20	-.313	20	.8751
		25	.8441

The first of these shows again that estimates for $y(0)$ decrease steadily with the step size, but the decrease is slow. The second also shows decrease, but it is *glacially* slow.

7. a. It is helpful to set $\ell(\ell+1) = L$. The various terms begin as follows:

$$\begin{aligned} y'' &= 2a_2 + 3 \cdot 2a_3x + 4 \cdot 3a_4x^2 + 5 \cdot 4a_5x^3 + 6 \cdot 5a_6x^4 \\ -x^2y'' &= -2a_2x^2 - 3 \cdot 2a_3x^3 - 4 \cdot 3a_4x^4 \\ -2xy' &= -2a_1x - 4a_2x^2 - 6a_3x^3 - 8a_4x^4 \\ Ly &= La_0 + La_1x + La_2x^2 + La_3x^3 + La_4x^4 \end{aligned}$$

Since the sum of the coefficients in each column must be zero, we find

$$\begin{aligned} 1 \cdot 2a_2 + La_0 &= 0 \\ 2 \cdot 3a_3 + (L-2)a_1 &= 0 \\ 3 \cdot 4a_4 + (L-2 \cdot 3)a_2 &= 0 \\ 4 \cdot 5a_5 + (L-3 \cdot 4)a_3 &= 0 \\ 5 \cdot 6a_6 + (L-4 \cdot 5)a_4 &= 0 \\ &\vdots \end{aligned}$$

and this implies

$$\begin{aligned} a_2 &= -\frac{L}{2}a_0 \\ a_3 &= -\frac{L-2}{2 \cdot 3}a_1 \\ a_4 &= -\frac{L-2 \cdot 3}{3 \cdot 4}a_2 = \frac{L(L-2 \cdot 3)}{4!}a_0 \\ a_5 &= -\frac{L-3 \cdot 4}{4 \cdot 5}a_3 = \frac{(L-1 \cdot 2)(L-3 \cdot 4)}{5!}a_1 \\ a_6 &= -\frac{L-4 \cdot 5}{5 \cdot 6}a_4 = -\frac{L(L-2 \cdot 3)(L-4 \cdot 5)}{6!}a_0 \end{aligned}$$

All the even terms depend on a_0 , all the odd on a_1 . The general terms can be expressed as follows:

$$\begin{aligned} a_{2n} &= (-1)^n \frac{L(L-6) \cdots (L-(2n-2)(2n-1))}{(2n)!} a_0 \\ a_{2n+1} &= (-1)^n \frac{(L-2)(L-12) \cdots (L-(2n-1)2n)}{(2n+1)!} a_1 \end{aligned}$$

The general solution to the differential equation is the following series; any choices of a_0 and a_1 give a solution.

$$\begin{aligned} &a_0 \left(1 - \frac{L}{2!}x^2 + \frac{L(L-6)}{4!}x^4 - \frac{L(L-6)(L-20)}{6!}x^6 + \cdots \right) \\ &+ a_1 \left(x - \frac{(L-2)}{3!}x^3 + \frac{(L-2)(L-12)}{5!}x^5 - \cdots \right) \\ &= a_0 \sum_{n=0}^{\infty} (-1)^n \frac{L(L-6) \cdots (L-(2n-2)(2n-1))}{(2n)!} x^{2n} \\ &+ a_1 \sum_{n=0}^{\infty} (-1)^n \frac{(L-2)(L-12) \cdots (L-(2n-1)2n)}{(2n+1)!} x^{2n+1} \end{aligned}$$

7. b. Each coefficient in the series is made up of factors of the form $L - k(k+1)$, where k is an integer. Therefore, such a factor will equal zero whenever $L = \ell(\ell+1)$ is equal to $k(k+1)$ for some k ; in other words, whenever ℓ equals the integer k . In fact, $L - k(k+1)$ first occurs in

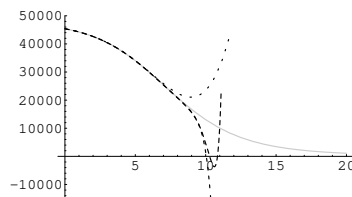
a_{k+2} , and then in every other coefficient: $a_{k+4}, a_{k+6}, \dots$. Here is a summary:

ℓ	L	terms that are zero
0	0	$a_2 = a_4 = a_6 = \dots$
1	2	$a_3 = a_5 = a_7 = \dots$
2	6	$a_4 = a_6 = a_8 = \dots$
3	12	$a_5 = a_7 = a_9 = \dots$
4	20	$a_6 = a_8 = a_{10} = \dots$
5	30	$a_7 = a_9 = a_{11} = \dots$

We get the following polynomial solutions:

ℓ	choose	polynomial solution
0	$a_1 = 0$	$y = a_0$
1	$a_0 = 0$	$y = a_1 x$
2	$a_1 = 0$	$y = a_0(1 - 3x^2)$
3	$a_0 = 0$	$y = a_1(x - \frac{5}{3}x^3)$
4	$a_1 = 0$	$y = a_0(1 - 10x^2 + \frac{35}{3}x^4)$
5	$a_0 = 0$	$y = a_1(x - \frac{14}{3}x^3 + \frac{21}{5}x^5)$

8. The text has carried out the work for the function $I(t)$, and found that the series *centered at $t = 0$* appears to converge for $t < 10$. We do a similar analysis for $S(t)$. The text provides the coefficients of the first 35 terms. The following are the graphs of P_9 (dots), P_{21} (short dashes), and P_{30} (longer dashes), together with a graph of S obtained by direct numerical integration (gray).



The graphs suggest that the series for $S(t)$ will converge only for $t < 10$.

9. a. To carry out the work here, we must first construct the series for y^2 from the series $y = a_0 + a_1x + a_2x^2 + \dots$.

$$\begin{aligned} y^2 &= (a_0 + a_1x + a_2x^2 + \cdots)(a_0 + a_1x + a_2x^2 + \cdots) \\ &= a_0^2 + 2a_0a_1x + (2a_0a_2 + a_1^2)x^2 \\ &\quad + (2a_0a_3 + 2a_1a_2)x^3 + (2a_0a_4 + 2a_1a_3 + a_2^2)x^4 \\ &\quad + (2a_0a_5 + 2a_1a_4 + 2a_2a_3)x^5 + \cdots \end{aligned}$$

Therefore,

$$\begin{aligned} y - y^2 &= (a_0 - a_0^2) + (a_1 - 2a_0a_1)x \\ &+ (a_2 - (2a_0a_2 + a_1^2))x^2 + (a_3 - (2a_0a_3 + 2a_1a_2))x^3 \\ &+ (a_4 - (2a_0a_4 + 2a_1a_3 + a_2^2))x^4 + \cdots \\ &= a_0(1 - a_0) + a_1(1 - 2a_0)x + (a_2(1 - 2a_0) - a_1^2)x^2 \\ &+ (a_3(1 - 2a_0) - 2a_1a_2)x^3 \\ &+ (a_4(1 - 2a_0) - 2a_1a_3 - a_2^2)x^4 + \cdots \end{aligned}$$

Since $y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots$, we get the recursion relations

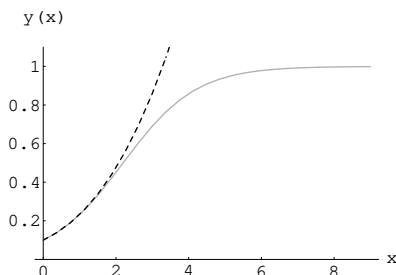
$$\begin{aligned} a_1 &= a_0(1 - a_0) \\ a_2 &= \frac{a_1(1 - 2a_0)}{2} \\ a_3 &= \frac{a_2(1 - 2a_0) - a_1^2}{3} \\ a_4 &= \frac{a_3(1 - 2a_0) - 2a_1a_2}{4} \end{aligned}$$

Here each a_n is expressed in terms of the various a_k for $k < n$, so each a_n ultimately can be expressed entirely in terms of a_0 . However, these expressions are unwieldy, and they aren't needed for the calculations we do in the next part.

9. b. The condition $y(0) = .1$ implies $a_0 = .1$. The recursion relations then give us

$$\begin{aligned} a_1 &= .09 \\ a_2 &= .036 \\ a_3 &= .0069 \\ a_4 &= -.00024 \end{aligned}$$

so the first five terms of the power series solution are $y = .1 + .09x + .036x^2 + .0069x^3 - .00024x^4$.

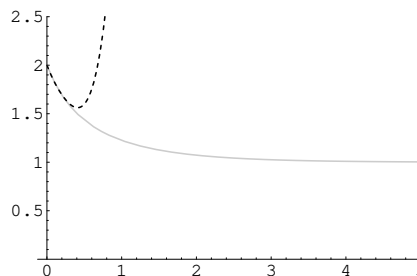


The graph of the first five terms of the power series solution and the direct numerical solution begin to differ visibly at $x \approx 1.8$.

9. c. If $y(0) = 2$, then $a_0 = 2$ and

$$\begin{aligned} a_1 &= -2 \\ a_2 &= 3 \\ a_3 &= -13/3 \\ a_4 &= 25/4 \end{aligned}$$

so the first five terms of the power series solution are $y = 2 - 2x + 3x^2 - \frac{13}{3}x^3 + \frac{25}{4}x^4$.



The graph of the first five terms of the power series solution and the direct numerical solution begin to differ visibly at $x \approx .4$.

10.5 Convergence

1. a. Use $1 - u + u^2 - u^3 + \cdots = 1/(1 + u)$ with $u = x^3$: the sum is $1/(1 + x^3)$.

1. b. $x^2 + x^6 + \cdots = x^2(1 + x^4 + \cdots) = x^2/(1 - x^4)$.

1. c. Let $f(x) = 1 - x + x^2 - x^3 + \cdots = 1/(1 + x)$. Then $f'(x) = -1 + 2x - 3x^2 + \cdots = -(1 - 2x + 3x^2 - \cdots)$ on the one hand, and $f'(x) = -(1 + x)^{-2}$ on the other. Therefore $1 - 2x + 3x^2 = \cdots = 1/(1 + x)^2$.

1. d. $x + 2x^2 + 3x^3 + \cdots = x(1 + 2x + 3x^2 + \cdots) = x(1 + x + x^2 + x^3 + \cdots)' = x \cdot (1/(1 - x))' = 1/(1 - x)^2$.

1. e. Let $f(x) = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$. Then $f'(x) = 1 + x + x^2 + \cdots = 1/(1 - x)$. Therefore $f(x)$ is an antiderivative of $1/(1 - x)$, so $f(x) = -\ln(1 - x) + C$ for some constant C . Since $f(0) = 0$, we have $C = 0$ so the sum of the series is exactly $-\ln(1 - x)$.

2. a. The series is

$$\frac{1}{4} \left(1 + \frac{1}{4} + \frac{1}{4^2} + \cdots \right) = \frac{1}{4} \cdot \frac{1}{1 - 1/4} = \frac{1}{3}.$$

2. b. The series is $\frac{2}{100} + \frac{2}{100^2} + \cdots =$

$$\frac{2}{100} \left(1 + \frac{1}{100} + \frac{1}{100^2} + \cdots \right) = \frac{2}{100} \cdot \frac{1}{1 - 1/100} = \frac{2}{99}.$$

2. c. The series is

$$-\frac{5}{2} \left(1 - \frac{1}{2} + \frac{1}{2}^2 - \cdots \right) = -\frac{5}{2} \cdot \frac{1}{1+1/2} = -\frac{5}{3}.$$

2. d. This series is just the series in exercise 1c with $x = 1/2$. Therefore its sum is $1/(1+x)^2 = 4/9$.

2. e. This series is just the series in exercise 1e with $x = 1/10$. Therefore its sum is $-\ln(1-.1) = -\ln(.9) = .10536\dots$ (Incidentally, this is what you get if you just start adding up the terms of the series, expressed as decimals.)

3. a. $S = A(1 + \frac{3}{5} + \frac{3}{5}^2 + \cdots) = A/(1 - \frac{3}{5}) = \frac{5}{2}A$.

3. b. The multiplier is $m = \frac{5}{2}$.

3. c. If the recirculation rate is r , then $S = A(1 + r + r^2 + \cdots) = A/(1 - r)$, so the multiplier in this case is $m = 1/(1 - r)$.

3. d. The multiplier is $m = 1/(1 - \frac{1}{5}) = \frac{5}{4}$.

3. e. Yes; if $r_1 < r_2$, then $1 - r_1 > 1 - r_2$ so $m_1 = 1/(1 - r_1) < 1/(1 - r_2) = m_2$.

4. Use the alternating series test in every case.

4. a. Converges, because $b_n \rightarrow 0$ as $n \rightarrow \infty$ and

$$\begin{aligned} b_{n+1} &= \frac{n+1}{(n+1)^2+1} \stackrel{?}{\leq} \frac{n}{n^2+1} = b_n \\ (n+1)(n^2+1) &\stackrel{?}{\leq} n(n^2+2n+2) \\ n^3+n^2+n+1 &\stackrel{?}{\leq} n^3+2n^2+2n \\ 1 &\stackrel{?}{\leq} n^2+n, \end{aligned}$$

and this is true for all $n \geq 1$.

4. b. Converges, because $b_n \rightarrow 0$ as $n \rightarrow \infty$ and $1/\sqrt{3(n+1)+2} < 1/\sqrt{3n+2}$.

4. c. Diverges, because $b_n \rightarrow \infty$, not 0, as $n \rightarrow \infty$.

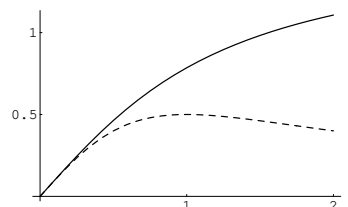
4. d. Diverges, because $b_n \rightarrow \frac{1}{5} \neq 0$ as $n \rightarrow \infty$.

4. e. Converges. First, $\arctan(n)/n < \pi/2n \rightarrow 0$ as $n \rightarrow \infty$. To show $b_{n+1} \leq b_n$, let $f(x) = \arctan(x)/x$. We will show $f'(x) < 0$ for $x > 1$. This will imply $b_{n+1} = f(n+1) < f(n) = b_n$. In fact,

$$f'(x) = \frac{x \cdot 1/(1+x^2) - \arctan(x)}{x^2},$$

so it remains to show $p(x) = x/(1+x^2) < \arctan(x) = q(x)$. In fact, $p(1) = 1/2 < \pi/4 = q(1)$. Furthermore,

$p'(x) = (1-x^2)/(1+x^2)^2 < 0$ when $x > 1$, so $p(x)$ is decreasing as $x \rightarrow \infty$. On the other hand, $q'(x) = 1/(1+x^2) > 0$, so $q(x)$ is increasing as $x \rightarrow \infty$. Therefore $p(x) < q(x)$ for all $x > 1$, proving that $f'(x) < 0$ for $x > 1$. In the following figure, $q(x)$ is the solid graph and $p(x)$ is the dashed.



4. f. Diverges, because we can show $b_n \rightarrow \infty$ as $n \rightarrow \infty$. Write $1.0001 = e^{\ln(1.0001)}$; then $(1.0001)^n = e^{\ln(1.0001)n} = e^{kn}$, where $k = \ln(1.0001) > 0$. But then, from chapter 4.3 (pages 236–237), we know that exponential functions grow more rapidly than any power function. Therefore,

$$b_n = \frac{(1.0001)^n}{n^{10}+1} = \frac{e^{kn}}{n^{10}+1} \rightarrow \infty \quad \text{as } n \rightarrow \infty,$$

which shows the series diverges.

4. g. Diverges. First, $n^{1/n} = e^{\ln(n)/n}$. Next, note that $\ln(n)/n \rightarrow 0$ as $n \rightarrow \infty$, so $e^{\ln(n)/n} \rightarrow e^0 = 1$. Therefore $b_n \rightarrow 1 \neq 0$ as $n \rightarrow \infty$.

4. h. Converges. First, $\ln(n) \rightarrow \infty$ as $n \rightarrow \infty$, so $b_n \rightarrow 0$. Next, $\ln(n) < \ln(n+1)$, so $b_{n+1} = 1/\ln(n+1) < 1/\ln(n) = b_n$.

4. i. Converges. It is relatively straightforward to show $b_{n+1} \leq b_n$:

$$\frac{(n+1)!}{(n+1)^{n+1}} = \frac{(n+1) \cdot n!}{(n+1) \cdot (n+1)^n} = \frac{n!}{(n+1)^n} < \frac{n!}{n^n}.$$

Showing $b_n \rightarrow 0$ is more difficult. With a computer we can calculate b_n for increasing n :

n	$n!/n^n$
100	9.3326×10^{-43}
1000	4.0239×10^{-433}
5000	5.9727×10^{-2170}

To prove the limit is zero, it is helpful to use Stirling's formula, chapter 12.1: $n! \sim cn^{n+\frac{1}{2}}e^{-n}$, where $c = \sqrt{2\pi}$.

$$\frac{n!}{n^n} \sim \frac{cn^n \sqrt{n} e^{-n}}{n^n} = \frac{c\sqrt{n}}{e^n} \rightarrow 0,$$

because exponential functions grow faster than any power.

4.j. Converges. Again, it is straightforward to show $b_{n+1} \leq b_n$:

$$\begin{aligned} b_{n+1} &= \frac{(n+1)!}{1 \cdot 3 \cdots (2n-1)(2n+1)} \\ &= \frac{n+1}{2n+1} \cdot \frac{n!}{1 \cdot 3 \cdots (2n-1)} \\ &< \frac{n!}{1 \cdot 3 \cdots (2n-1)} = b_n. \end{aligned}$$

We have used the fact that $(n+1)/(2n+1) < 1$ for $n \geq 2$. Computation suggests that $b_n \rightarrow 0$:

n	$n!/(1 \cdot 3 \cdot 5 \cdots (2n-1))$
100	1.4000×10^{-29}
500	1.2111×10^{-149}
1000	5.2316×10^{-300}

To prove $b_n \rightarrow 0$, we first express b_n in terms of factorial functions:

$$\begin{aligned} \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} &= \frac{n! \cdot 2 \cdot 4 \cdots 2n}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdots 2n} \\ &= \frac{n! \cdot 2^n \cdot n!}{(2n)!} = \frac{2^n (n!)^2}{(2n)!} \end{aligned}$$

and then analyze this using Stirling's formula:

$$\begin{aligned} \frac{2^n \cdot (n!)^2}{(2n)!} &= \frac{2^n \cdot c^2 n^{2n} n e^{-2n}}{c(2n)^{2n} \sqrt{2n} e^{-2n}} \\ &= \frac{c^2 n^{2n} n}{2^{2n} \sqrt{2n}^{2n} \sqrt{n}} = \frac{c\sqrt{n}}{2^n \sqrt{2}} \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$ (because exponential functions grow faster than powers).

5. Since these series are all *alternating*, the partial sum S_n will be correct to six decimal places if $b_{n+1} \leq .5 \times 10^{-6}$ (according to the criterion on page 651).

5.a. Since $b_{n+1} = \frac{n+1}{(n+1)^2 + 1} < \frac{1}{n+1}$, it is sufficient to have $\frac{1}{n+1} < .5 \times 10^{-6}$. This will be true if $n = 2 \times 10^6$. We can now find the partial sum $S_{2 \times 10^6}$ with a program like the following one in QuickBasic:

```
DEFDBL A-Z
N = 2 * 10 ^ 6
sign = -1
sum = 0
FOR k = 1 TO N
    sum = sum + sign * k / (k ^ 2 + 1)
    sign = -1 * sign
NEXT k
PRINT N, Sum
```

The result is $S_n = -.2696102527 \dots$. The full infinite sum lies between this value and

$$S_{n+1} = S_n - b_{n+1} > -.2696103527 \dots;$$

to six decimal places, this is $-.269610$.

5.b. $1/\sqrt{3(n+1)+2} \leq .5 \times 10^{-6}$ if $\sqrt{3n+2} \geq 2 \times 10^6$, so $3n+2 \geq 4 \times 10^{12}$. It is sufficient to take $n = 1.5 \times 10^{12}$. *This series converges very slowly indeed.* Using a modification of the Basic program above on a 486 machine it took about 10 minutes to compute $S_{1.5 \times 10^7} = -.25440388 \dots$, and this is accurate only to three decimal places. (In fact, $S_{1+1.5 \times 10^7} = -.25455295 \dots$) To compute $S_{1.5 \times 10^{12}}$ would take 10^5 times as long, or about two years! We can at least say $S = -.254 \dots$

5.e. Since $\arctan(n) < \pi/2$, we have

$$\arctan(n+1)/(n+1) < \pi/2(n+1) \leq .5 \times 10^{-6}.$$

This implies $n+1 \geq \pi \times 10^6$, and this will be true if we take $n = 4 \times 10^6$. With a suitable Basic program we find $S_{4 \times 10^6} = -.465712107 \dots$, so the full infinite sum is $-.465712 \dots$

5.h. $1/\ln(n+1) \leq .5 \times 10^{-6}$ if $\ln(n+1) \geq 2 \times 10^6$, or $n+1 \geq \exp(2 \times 10^6) = 9.2 \times 10^{868588}$. Thus, to get six decimal places accuracy, we would need to take $n = 10^{868589}$. This would take a Basic program running on a 486 machine much more than a human lifetime to compute! We give no value for this sum.

5.i. This series converges rather quickly, as we can see by evaluating $n!/n^n$ for a few small values of n :

n	$n!/n^n$
5	0.0384
10	3.6288×10^{-4}
15	2.9863×10^{-6}
16	1.1342×10^{-6}
17	0.4300×10^{-6}

Thus $b_{17} < .5 \times 10^{-6}$, so S_{16} will be accurate to 6 decimal places. Now $S_{16} = -.655831288 \dots$, so the infinite sum is $-.655831 \dots$

5.j. Again, calculate the size of the first few terms, using the formula $b_n = 2^n (n!)^2 / (2n)!$ constructed in the last exercise.

n	$2^n (n!)^2 / (2n)!$
5	0.1270
10	5.5425×10^{-2}
15	2.1145×10^{-4}
20	7.6068×10^{-6}
24	0.5203×10^{-6}
25	0.2654×10^{-6}

Thus $S_{24} = -.58678182\dots$ is accurate to 6 decimal places, so the infinite sum is $-.586781\dots$

6. Since $-3 \leq x \leq 0$, the Taylor series for e^x is an *alternating* series (for each fixed x). Therefore, we can use the alternating series criterion we used in the previous exercise, and determine when $b_n = |x|^n/n! \leq .5 \times 10^{-10}$. Of course this depends upon x ; however, the “worst case” occurs when $x = -3$. In other words, we want to find n so that $3^n/n! \leq .5 \times 10^{-10}$, and we can do this by trial and error:

n	$3^n/n!$
5	2.025
10	1.6272×10^{-2}
15	1.0973×10^{-5}
20	1.4332×10^{-9}
21	2.0474×10^{-10}
22	0.2792×10^{-10}

Thus the Taylor polynomial of degree 21 will be accurate to 10 decimal places. (In fact, we find that, for the worst case,

$$P_{21}(-3) = .049787068343\dots$$

$$e^{-3} = .049787068367\dots,$$

so the two *do* agree to 10 decimal places.)

7. a.	n	S_n	n	S_n
	1	1	7	2.5928571429
	2	1.5	8	2.7178571429
	3	1.8333333333	9	2.8289682540
	4	2.0833333333	10	2.9289682540
	5	2.2833333333	11	3.0198773449
	6	2.45	12	3.1032106782

7. b.	n	S_n
	100	5.1873775176
	500	6.7928234300
	1000	7.4854708606
	5000	9.0945088530

7. c. The second parentheses contains 2 terms and each is at least $1/2^2$, so their sum is at least $2 \times 1/2^2 = 1/2$. The third parentheses contains 2^2 terms and each is at least $1/2^3$, so their sum is at least $2^2 \times 1/2^3 = 1/2$. The general pattern is this: for each $k = 1, 2, 3, \dots$, the k th parentheses contains 2^{k-1} terms and each is at least $1/2^k$, so their sum is at least $2^{k-1} \times 1/2^k = 1/2$.

7. d. If $n = 2^k$, then

$$S_n = 1 + [k \text{ groups, each with sum } \geq 1/2]$$

$$\geq 1 + k \cdot \frac{1}{2}$$

7. e. We show that the series diverges by showing $\lim_{n \rightarrow \infty} = \infty$. Specifically, for any given M , we show there

is an n for which $S_n > M$. In fact, since $S_{2^k} > 1 + \frac{k}{2}$, just choose $k = 2M$. Then $1 + \frac{k}{2} > M$, so certainly $S_{2^k} > M$.

7. f. We measure distance leftward from the right edge of the top card. Let x_n be the position of the right edge of the n th card. then $x_1 = 0$ and $x_2 = \frac{1}{2}$. We want to show $x_{n+1} = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n}$. Equivalently, we can try to show $x_{n+1} - x_n = \frac{1}{2^n}$.

We need to assemble some facts.

(1) The cards are stacked so that the combined center of mass C_{n-1} of the top $n-1$ cards lies directly above the right edge x_n of the n th card: $C_{n-1} = x_n$.

(2) The individual cards have mass 1.

(3) If c_n denotes the center of mass of the n th card, then $c_n = x_n + \frac{1}{2}$.

(4) We have the general principle: if mass m_1 has center of mass q_1 and m_2 has center of mass q_2 (both on the x -axis) then their *combined* center of mass Q satisfies the equation $MQ = m_1q_1 + m_2q_2$ where $M = m_1 + m_2$.

Now view the top n cards (total mass n) to be made up of the top $n-1$ cards (total mass $n-1$) plus the n th card (mass 1). Then, according to the general principle (4),

$$nC_n = (n-1)C_{n-1} + c_n.$$

Now replace C_n , C_{n-1} , and c_n using (1) and (3):

$$nx_{n+1} = (n-1)x_n + x_n + \frac{1}{2} = nx_n + \frac{1}{2}$$

It now follows immediately that $x_{n+1} - x_n = \frac{1}{2^n}$.

8. a. The graph of $y = 1/x$ lies above the dark-shaded rectangles but within the combined light- and dark-shaded rectangles. Therefore the area under the graph is more than the area of the dark-shaded rectangles but less than the area of the combined rectangles. The area of the dark-shaded rectangles is $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = S_6 - 1$. Since the area under the graph is the integral given in the problem, we have

$$S_6 - 1 < \int_1^6 \frac{1}{x} dx = \ln(6) < S_6 - \frac{1}{6}.$$

A bit of algebra converts this into

$$\frac{1}{6} < S_6 - \ln(6) < 1.$$

8. b. Replace $x = 6$ in the previous argument by $x = n$. Then the area of the dark-shaded rectangles is $S_n - 1$, while the area of the combined rectangles is $S_{n-1} = S_n - \frac{1}{n}$. since the area under the graph is the integral of $1/x$ from $x = 1$ to $x = n$, we get

$$S_n - 1 < \int_1^n \frac{1}{x} dx = \ln(n) < S_n - \frac{1}{n},$$

and hence

$$\frac{1}{n} < S_n - \ln(n) < 1.$$

8. c. From part b we have $.0001 < S_{10000} - \ln(10000) < 1$. Since $\ln(10000) = 9.21034\dots$, S_{10000} lies between 9.21044 and 10.21035.

8. d. Since n cards can extend to a (maximum!) distance of $\frac{1}{2}S_n$, to make the stack extend 100 units we need to find n for which $\frac{1}{2}S_n \geq 100$, or $S_n \geq 200$.

From part b we see that $\ln(n) + \frac{1}{n} < S_n$. therefore, we will have $200 \leq S_n$ if we choose n so that $200 \leq \ln(n)$, or $e^{200} \leq n$. Thus we need about $e^{200} \approx 7.23 \times 10^{86}$ cards!

[Note: If we use ordinary playing cards, each deck makes a stack about 1 centimeter high. Thus 100 decks (about 5000 cards) will make a stack 1 meter high, so 5×10^6 cards will make a stack about 1 kilometer high. To stack *all* the cards, we would need $(7 \times 10^{86}) / (5 \times 10^6) \approx 10^{80}$ km. This is a large distance; the diameter of the solar system is on the order of 10^{10} km. while the distance to the nearest star is on the order of 10^{15} km. The diameter of the entire universe is only about 10^{26} km. It's not clear we would have enough room to stack the cards—even if there were enough matter in the universe to make them!]

9. It follows from the discussion in the text that the radius of convergence of the power series $\sum a_n x^n$ is $R = \lim_{n \rightarrow \infty} |a_n|/|a_{n+1}|$ when this limit exists.

Arctangent. $\arctan(x) = x - x^3/3 + x^5/5 - \dots = x(1 - u/3 + u^2/5 - \dots \pm u^n/(2n+1) + \dots)$, where $u = x^2$. According to the ratio test, the radius of convergence of the last series is

$$R = \lim_{n \rightarrow \infty} \frac{1/(2n+1)}{1/(2n+3)} = \lim_{n \rightarrow \infty} \frac{2n+3}{2n+1} = 1.$$

Thus the series diverges for $|u| = |x^2| > 1$ and hence for $|x| > 1$.

If $x = \pm 1$, the series for the arctangent is alternating with terms that decrease to 0. Therefore the series converges by the alternating series test.

The domain of convergence of the series is precisely $-1 \leq x \leq 1$.

Binomial series. The series for $(1+x)^c$ is $1 + cx + c(c-1)x^2/2! + c(c-1)(c-2)x^3/3! + \dots$. We can assume c is not 0 or a positive integer (for otherwise the series collapses to a finite sum and there is no issue of convergence).

The radius of convergence is

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \frac{|c(c-1) \cdots (c-(n-1))|/n!}{|c(c-1) \cdots (c-(n-1))(c-n)|/(n+1)!} \\ &= \lim_{n \rightarrow \infty} \frac{n+1}{|c-n|} \\ &= \lim_{n \rightarrow \infty} \frac{1+1/n}{1-c/n} = 1 \end{aligned}$$

Therefore the series diverges for $|x| > 1$.

Now let $x = 1$. We'll show that the series converges when $-1 < c$ and diverges otherwise. Since c is fixed and $n \rightarrow \infty$, n is eventually larger than c . When that happens, each of the factors $c-n$ is negative, so the product $c(c-1) \cdots (c-(n-1))$ alternates in sign as n increases. Therefore the series is (eventually) an alternating series. We can use the alternating series test to say that the series converges if we can show that the coefficients are decreasing and approach 0 as $n \rightarrow \infty$.

$$\begin{aligned} b_{n+1} &= \frac{|c(c-1) \cdots (c-(n-1))(c-n)|}{(n+1)!} \\ &= \frac{|c-n|}{n+1} \cdot \frac{|c(c-1) \cdots (c-(n-1))|}{n!} \\ &= \frac{n-c}{n+1} \cdot b_n \end{aligned}$$

Here we assume n is large enough ($n > c$ will do) to allow us to set $|c-n| = n-c$. It follows that $b_{n+1} \geq b_n$ precisely when $n-c \geq n+1$, or $c \leq -1$ (and n is sufficiently large). In this case it is impossible for $b_n \rightarrow 0$ as $n \rightarrow \infty$, so the series diverges when $c \leq -1$.

If $c > -1$, we can show $b_n \rightarrow 0$ as $n \rightarrow \infty$. We'll first deal with the case when $c > 0$. (This argument doesn't work when $-1 < c < 0$.) Now let K be the smallest integer that is greater than c . In other words, $0 < K-c < 1$ and K is at least 1 (because $c > 0$). Then

$$\begin{aligned} 0 \leq b_n &= \frac{|c(c-1) \cdots (c-(n-1))|}{n!} \\ &= \frac{|c||c-1| \cdots |c-(n-1)|}{n!} \\ &= M \frac{(K+1-c) \cdots (n-1-c)}{n!} \end{aligned}$$

where $M = |c(c-1) \cdots (c-K)| \neq 0$. Since $0 < K-c < 1$, we have $K+1-c < 2$, $K+2-c < 3$, and so forth. Hence

$$(K+1-c)(K+2-c) \cdots (n-1-c) \leq 2 \cdot 3 \cdots (n-K-1)$$

so

$$b_n \leq M \frac{(n-K)!}{n!} \leq \frac{M}{n} \rightarrow 0$$

as $n \rightarrow \infty$. (Note that we need $K \geq 1$ here to ensure that $(n-K)!/n! \leq 1/n$. If $K = 0$ the numerator is $n!$ so the fraction does not go to 0 as $n \rightarrow \infty$.) All the conditions of the alternating series test are met, so the series does indeed converge for $x = 1$ when $c > 0$.

Probably the best thing to do when $-1 < c < 0$ is to compute some values. For example, it appears that when $c = -1/2$ the series converges to $1/\sqrt{2}$ (as it should), but the convergence is quite slow.

When $x = -1$, the series converges precisely when $0 \leq c$. This is plausible because the expression the series would converge to is $(1-1)^c = 0^c$, and this is finite precisely when $0 \leq c$.

To justify the claim, note first that the powers of x alternate in sign. Since the coefficients (eventually) alternate in sign, these two alternations cancel. Eventually, all terms have the same sign, so the alternating series test is of no avail.

Again, there is no easy proof. The simplest course of action is to estimate the value of the series with a computer for various values of c . It appears that when $c > 0$ the sum always converges to 0.

10. We use the general fact that the radius of convergence of the series $\sum a_n x^n$ is $R = \lim_{n \rightarrow \infty} |a_n/a_{n+1}|$ when this limit exists.

10. a. $a_n = (n+1)$, so $R = \lim_{n \rightarrow \infty} \frac{n+1}{n+2} = 1$

10. b. $a_n = n$, so $R = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$.

10. c. $a_n = 1/n^2$ (for $n \geq 1$), so

$$\begin{aligned} R &= \lim_{n \rightarrow \infty} \frac{1/n^2}{1/(n+1)^2} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^2 \\ &= \lim_{n \rightarrow \infty} (1 + 1/n)^2 = 1. \end{aligned}$$

10. d. Let $u = x^3$; then we can write the given series as $u(1 + u + u^2 + \dots)$. This is a geometric series whose radius of convergence is 1; that is, $|u| < 1$. But since $u = x^3$, we also have $|x| < 1$.

10. e. Let $u = x + 1$; the series is $1 + u + u^2 + \dots$, a geometric series which converges for $|u| < 1$. The original series therefore converges for $|x+1| < 1$, so $R = 1$ here as well.

10. f. Ignore the first term, since it has no effect on convergence. In the remaining series we can take $a_n = 1/3^n$, so so $R = \lim_{n \rightarrow \infty} \frac{1/3^n}{1/3^{n+1}} = \lim_{n \rightarrow \infty} 3 = 3$

11. a. $0 + x + 2x^2 + 3x^3 + 4x^4 + \dots$. This is the same series as in exercise 10b, so $R = 1$

11. b. $0 + \frac{1}{2}x + x^2 + \frac{9}{8}x^3 + x^4 + \frac{25}{32}x^5 + \dots$. Here

$$R = \lim_{n \rightarrow \infty} \frac{n^2/2^n}{(n+1)^2/2^{n+1}} = \lim_{n \rightarrow \infty} 2(n/(n+1))^2 = 2.$$

11. c. $25 + 36x + 49x^2 + 64x^3 + 81x^4 + \dots$. Here

$$R = \lim_{n \rightarrow \infty} \frac{(n+5)^2}{(n+6)^2} = \lim_{n \rightarrow \infty} \left(\frac{n+5}{n+6} \right)^2 = 1.$$

11. d. Since $n^n = e^{n \ln(n)}$, and since $x \ln(x) \rightarrow 0$ as $x \rightarrow 0$ (exercise 32 of section 10.2), we are justified in defining $n^n = e^0 = 1$ when $n = 0$. Therefore the first term of the series in this question is 99, and the first five terms are $99 + 99x + \frac{99}{4}x^2 + \frac{99}{27}x^3 + \frac{99}{256}x^4 + \dots$. Then

$$R = \lim_{n \rightarrow \infty} \frac{99/n^n}{99/(n+1)^{n+1}} = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^n \cdot (n+1).$$

Now recall that $\lim_{n \rightarrow \infty} \left(\frac{n+1}{n} \right)^n = \lim_{n \rightarrow \infty} (1 + 1/n)^n = e$ (text page 230), but since $\lim_{n \rightarrow \infty} (n+1) = \infty$, it follows that $R = \infty$.

11. e. Since $0! = 1$ (text page 615), the first five terms are $1 + x + 2x^2 + 6x^3 + 24x^4 + \dots$. Hence $R = \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = \lim_{n \rightarrow \infty} 1/(n+1) = 0$.

12. The general term of the Taylor series for $\sin(x)$ is $b_m = (-1)^m x^{2m+1}/(2m+1)!$. According to the ratio test, the series converges if

$$\begin{aligned} \lim_{m \rightarrow \infty} \left| \frac{b_{m+1}}{b_m} \right| &= \lim_{m \rightarrow \infty} \frac{|x|^{2m+3}/(2m+3)!}{|x|^{2m+1}/(2m+1)!} \\ &= \lim_{m \rightarrow \infty} \frac{|x|^2}{(2m+2)(2m+3)} < 1 \end{aligned}$$

But $\lim_{m \rightarrow \infty} |x|^2/(2m+2)(2m+3) = 0$ for all x , so $R = \infty$.

The general term of the Taylor series for $\cos(x)$ is $b_m = (-1)^m x^{2m}/(2m)!$. This time the ratio test gives

$$\begin{aligned} \lim_{m \rightarrow \infty} \left| \frac{b_{m+1}}{b_m} \right| &= \lim_{m \rightarrow \infty} \frac{|x|^{2m+2}/(2m+2)!}{|x|^{2m}/(2m)!} \\ &= \lim_{m \rightarrow \infty} \frac{|x|^2}{(2m+1)(2m+2)} < 1 \end{aligned}$$

and this limit is also 0, independent of the value of x . Therefore $R = \infty$ here as well.

The Taylor series for $\sin(x)$ and for $\cos(x)$ can therefore represent those functions for all x .

13. The series is $1 - x^2 + x^4 - x^6 + \dots = \sum_{m=0}^{\infty} (-1)^m x^{2m}$.

Since this is just a geometric series $1 + r + r^2 + \dots$ in which the multiplier $r = -x^2$, the radius of convergence

is 1. Therefore the series can represent the function only for $-1 < x < 1$. However, the function itself (that is, $1/(1+x^2)$) is defined for *all* real numbers x .

14. a. If $g(x) = 1/f(x)$, then, by the quotient rule,

$$g'(x) = -\frac{f'(x)}{(f(x))^2}.$$

14. b. $\Delta g \approx g'(x)\Delta x$, and

$$\begin{aligned} \text{relative error in } g &= \frac{\Delta g}{g} \\ &\approx \frac{g'(x)\Delta x}{g(x)} \\ &= \frac{-f'(x)/(f(x))^2\Delta x}{1/f(x)} = -\frac{f'(x)\Delta x}{f(x)} \\ &\approx -\text{relative error in } f \end{aligned}$$

In other words, the relative errors of f and $g = 1/f$ are (approximately) the same size but opposite in sign.

14. c. If $f = e^{-x}$ and $g = e^x$, then $g = 1/f$ so the preceding discussion applies. As noted in the text, $e \approx 7 \cdot e^{-1}$. We use this value in what follows. By part b, relative errors for $f(x)$ and $g(x)$ are the same size:

$$\frac{\Delta g}{g} \approx \pm \frac{\Delta f}{f}.$$

However, when $x = 1$, $g \approx 7f$, so

$$\Delta g \approx \pm g \cdot \frac{\Delta f}{f} \approx \pm 7\Delta f.$$

That is, the *actual* error in $g = e$ is about 7 times as large as the *actual* error in $f = e^{-1}$. Therefore, when the text got an estimate for e^{-1} that was accurate to seven decimal places, that means

$$|\Delta f| = \text{error in } e^{-1} < .5 \times 10^7.$$

But then

$$\begin{aligned} \text{error in } e &= |\Delta g| < 7 \times .5 \times 10^{-7} \\ &= 3.5 \times 10^{-7} < .5 \times 10^{-6}. \end{aligned}$$

This is how we lose one decimal place of accuracy in calculating e as the reciprocal of e^{-1} .

10.6 Approximation Over Intervals

1. a. $I_0 = \int_0^\pi \sin(x) dx = -\cos(x) \Big|_0^\pi = 2.$

$$\begin{aligned} I_1 &= \int_0^\pi x \sin(x) dx \\ &= -x \cos(x) \Big|_0^\pi + \int_0^\pi \cos(x) dx \\ &= -x \cos(x) + \sin(x) \Big|_0^\pi = \pi \end{aligned}$$

We used integration by parts (text, page 709) on the second line.

1. b. If we integrate $\int x^{n+2} \sin(x) dx$ by parts with $F = x^{n+2}$, $g = \sin(x)$ (and thus $f = (n+2)x^{n+1}$ and $G = -\cos(x)$), we get

$$\int x^{n+2} \sin(x) dx = -x^{n+2} \cos(x) + (n+2) \int x^{n+1} \cos(x) dx$$

If we integrate the new integral by parts with $F = x^{n+1}$ and $g = \cos(x)$ (so $f = (n+1)x^n$ and $G = \sin(x)$), we get

$$\int x^{n+1} \cos(x) dx = x^{n+1} \sin(x) - (n+1) \int x^n \sin(x) dx$$

Combining these results, we get

$$\begin{aligned} I_{n+2} &= \int_0^\pi x^{n+2} \sin(x) dx \\ &= -x^{n+2} \cos(x) + (n+2)x^{n+1} \sin(x) \Big|_0^\pi \\ &\quad - (n+2)(n+1) \int_0^\pi x^n \sin(x) dx \\ &= \pi^{n+2} - (n+2)(n+1)I_n \end{aligned}$$

1. c. We use the recursion formula just established:

$$\begin{aligned} I_2 &= \pi^2 - 2I_0 = \pi^2 - 4 \\ I_3 &= \pi^3 - 6I_1 = \pi^3 - 6\pi \\ I_4 &= \pi^4 - 12I_2 = \pi^4 - 12\pi^2 + 48 \end{aligned}$$

1. d. The idea is described on page 669 of the text. There, three equations in the three unknowns a_0 , a_1 , and a_2 are introduced. We must expand this to five equations in five

unknowns, as follows:

$$\begin{aligned} \int_0^\pi \sin(x) dx \\ &= \int_0^\pi (a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4) dx \\ &\vdots \\ \int_0^\pi x^4 \sin(x) dx \\ &= \int_0^\pi (a_0x^4 + a_1x^5 + a_2x^6 + a_3x^7 + a_4x^8) dx \end{aligned}$$

The left-hand sides are $I_0, \dots, I_4$, which we have just determined. The equations to be solved are therefore

$$\begin{aligned} 2 &= \pi a_0 + \frac{\pi^2}{2} a_1 + \frac{\pi^3}{3} a_2 + \frac{\pi^4}{4} a_3 + \frac{\pi^5}{5} a_4 \\ \pi &= \frac{\pi^2}{2} a_0 + \frac{\pi^3}{3} a_1 + \frac{\pi^4}{4} a_2 + \frac{\pi^5}{5} a_3 + \frac{\pi^6}{6} a_4 \\ \pi^2 - 4 &= \frac{\pi^3}{3} a_0 + \frac{\pi^4}{4} a_1 + \frac{\pi^5}{5} a_2 + \frac{\pi^6}{6} a_3 + \frac{\pi^7}{7} a_4 \\ \pi^3 - 6\pi &= \frac{\pi^4}{4} a_0 + \frac{\pi^5}{5} a_1 + \frac{\pi^6}{6} a_2 + \frac{\pi^7}{7} a_3 + \frac{\pi^8}{8} a_4 \\ \pi^4 - 12\pi^2 + 48 &= \frac{\pi^5}{5} a_0 + \frac{\pi^6}{6} a_1 + \frac{\pi^7}{7} a_2 + \frac{\pi^8}{8} a_3 + \frac{\pi^9}{9} a_4 \end{aligned}$$

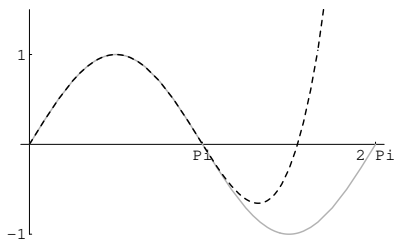
Mathematica's `Solve` command, applied to this system of equations, yields

$$\begin{aligned} a_0 &= .00131346 & a_1 &= .982601 \\ a_2 &= .0544697 & a_3 &= -.233793 \\ a_4 &= .0372093 \end{aligned}$$

We can therefore write the approximating polynomial as

$$Q(x) = .00131 + .98260x + .05447x^2 - .23379x^3 + .03721x^4.$$

We can find the maximum difference between $Q(x)$ and $\sin(x)$ by graphing $Q(x) - \sin(x)$. The graph below shows that the maximum difference occurs at the ends of the interval, specifically at $x = \pi$: $Q(\pi) - \sin(\pi) = .001471$. Since $.001471 < .5 \times 10^{-2}$, $Q(x)$ agrees with $\sin(x)$ to two decimal places over the entire interval $0 \leq x \leq \pi$.



2. a. Since $|x| = x$ when $0 \leq x$ and $|x| = -x$ when $x \leq 0$, we can write

$$I_n = \int_{-1}^1 x^n |x| dx = - \int_{-1}^0 x^{n+1} dx + \int_0^1 x^{n+1} dx.$$

Even and odd powers of x satisfy the following conditions:

$$- \int_{-1}^0 x^{n+1} dx = \begin{cases} - \int_0^1 x^{n+1} dx & \text{if } n+1 \text{ is even} \\ \int_0^1 x^{n+1} dx & \text{if } n+1 \text{ is odd} \end{cases}$$

Since n is odd when $n+1$ is even, and vice versa, it follows that

$$I_n = \int_{-1}^1 x^n |x| dx = - \int_0^1 x^{n+1} dx + \int_0^1 x^{n+1} dx = 0$$

if n is odd, while

$$= 2 \int_0^1 x^{n+1} dx = 2 \cdot \frac{x^{n+2}}{n+2} \Big|_0^1 = \frac{2}{n+2}$$

if n is even.

2. b. If $Q(x) = a_0 + a_1x + a_2x^2$ is the quadratic least squares approximation $Q(x)$ to $|x|$ over $[-1, 1]$, then

$$\begin{aligned} \int_{-1}^1 |x| dx &= \int_{-1}^1 (a_0 + a_1x + a_2x^2) dx \\ \int_{-1}^1 x|x| dx &= \int_{-1}^1 (a_0x + a_1x^2 + a_2x^3) dx \\ \int_{-1}^1 x^2|x| dx &= \int_{-1}^1 (a_0x^2 + a_1x^3 + a_2x^4) dx \end{aligned}$$

We have just seen that the values of the three integrals on the left are $I_0 = 1$, $I_1 = 0$, and $I_2 = 1/2$, respectively. When we evaluate the integrals on the right, the equations become

$$\begin{aligned} 1 &= 2a_0 + \frac{2}{3}a_2 \\ 0 &= \frac{2}{3}a_1 \\ \frac{1}{2} &= \frac{2}{3}a_0 + \frac{2}{5}a_2 \end{aligned}$$

These can be solved with bare hands: $a_0 = \frac{3}{16}$, $a_1 = 0$, $a_2 = \frac{15}{16}$. Therefore

$$Q(x) = \frac{3}{16} + \frac{15}{16}x^2 = .1875 + .9375x^2.$$

This agrees with the text.

2. c. Let $Q(x) = a_0 + \cdots + a_{10}x^{10}$ denote the tenth degree polynomial. We have eleven integral equations

$$\begin{aligned}\int_{-1}^1 |x| dx &= \int_{-1}^1 (a_0 + \cdots + a_{10}x^{10}) dx \\ \int_{-1}^1 x|x| dx &= \int_{-1}^1 (a_0x + \cdots + a_{10}x^{11}) dx \\ &\vdots \\ \int_{-1}^1 x^{10}|x| dx &= \int_{-1}^1 (a_0x^{10} + \cdots + a_{10}x^{20}) dx\end{aligned}$$

The left-hand sides are the I_n ; the right-hand sides integrate immediately to give these six equations involving the even coefficients:

$$\begin{aligned}1 &= 2a_0 + \frac{2}{3}a_2 + \frac{2}{5}a_4 + \frac{2}{7}a_6 + \frac{2}{9}a_8 + \frac{2}{11}a_{10} \\ \frac{1}{2} &= \frac{2}{3}a_0 + \frac{2}{5}a_2 + \frac{2}{7}a_4 + \frac{2}{9}a_6 + \frac{2}{11}a_8 + \frac{2}{13}a_{10} \\ \frac{1}{3} &= \frac{2}{5}a_0 + \frac{2}{7}a_2 + \frac{2}{9}a_4 + \frac{2}{11}a_6 + \frac{2}{13}a_8 + \frac{2}{15}a_{10} \\ \frac{1}{4} &= \frac{2}{7}a_0 + \frac{2}{9}a_2 + \frac{2}{11}a_4 + \frac{2}{13}a_6 + \frac{2}{15}a_8 + \frac{2}{17}a_{10} \\ \frac{1}{5} &= \frac{2}{9}a_0 + \frac{2}{11}a_2 + \frac{2}{13}a_4 + \frac{2}{15}a_6 + \frac{2}{17}a_8 + \frac{2}{19}a_{10} \\ \frac{1}{6} &= \frac{2}{11}a_0 + \frac{2}{13}a_2 + \frac{2}{15}a_4 + \frac{2}{17}a_6 + \frac{2}{19}a_8 + \frac{2}{21}a_{10}\end{aligned}$$

and these five involving the odd coefficients:

$$\begin{aligned}0 &= \frac{2}{3}a_1 + \frac{2}{5}a_3 + \frac{2}{7}a_5 + \frac{2}{9}a_7 + \frac{2}{11}a_9 \\ 0 &= \frac{2}{5}a_1 + \frac{2}{7}a_3 + \frac{2}{9}a_5 + \frac{2}{11}a_7 + \frac{2}{13}a_9 \\ 0 &= \frac{2}{7}a_1 + \frac{2}{9}a_3 + \frac{2}{11}a_5 + \frac{2}{13}a_7 + \frac{2}{15}a_9 \\ 0 &= \frac{2}{9}a_1 + \frac{2}{11}a_3 + \frac{2}{13}a_5 + \frac{2}{15}a_7 + \frac{2}{17}a_9 \\ 0 &= \frac{2}{11}a_1 + \frac{2}{13}a_3 + \frac{2}{15}a_5 + \frac{2}{17}a_7 + \frac{2}{19}a_9\end{aligned}$$

The second set of equations implies $a_1 = a_3 = a_5 = a_7 = a_9 = 0$; the polynomial has only *even* powers (just as do the ones in the text).

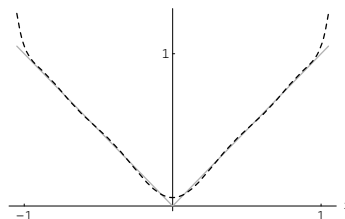
Mathematica's `Solve` command gives us the following for the even coefficients:

$$\begin{aligned}a_0 &= \frac{14553}{262144} & a_2 &= \frac{945945}{262144} \\ a_4 &= -\frac{1576575}{131072} & a_6 &= \frac{3216213}{131072} \\ a_8 &= -\frac{6235515}{262144} & a_{10} &= \frac{2263261}{262144}\end{aligned}$$

Thus

$$\begin{aligned}Q(x) &= .0555153 + 3.608494x^2 - 12.028313x^4 \\ &\quad + 24.537758x^6 - 23.786602x^8 + 8.663656x^{10}.\end{aligned}$$

The maximum difference between $Q(x)$ and $|x|$ occurs at the origin: $Q(0) = .0555153$.



3. a. $I_0 = \int_0^2 \frac{1}{1+x^2} = \arctan(x) \Big|_0^2 = \arctan(2) \approx 1.10715$. The other integral, I_1 , is quite readily found by substitution. Let $u = 1 + x^2$:

$$\int \frac{x}{1+x^2} dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln(u) = \frac{1}{2} \ln(1+x^2)$$

Thus $I_1 = \frac{1}{2} \ln(1+x^2) \Big|_0^2 = \frac{1}{2} \ln(5)$.

3. b. A bit of algebra takes care of this; neither integration by parts nor substitution is needed. The integrand of I_2 is

$$\frac{x^2}{1+x^2} = \frac{1+x^2-1}{1+x^2} = 1 - \frac{1}{1+x^2},$$

so it reduces to a simple expression involving the integrand of I_0 . More generally,

$$\frac{x^{n+2}}{1+x^2} = x^n \frac{1+x^2-1}{1+x^2} = x^n - \frac{x^n}{1+x^2}$$

Therefore

$$\begin{aligned}I_{n+2} &= \int_0^2 \frac{x^{n+2}}{1+x^2} dx = \int_0^2 x^n dx - \int_0^2 \frac{x^n}{1+x^2} dx \\ &= \frac{x^{n+1}}{n+1} \Big|_0^2 - I_n = \frac{2^{n+1}}{n+1} - I_n\end{aligned}$$

3. c. From the recursion formula just obtained we find

$$\begin{aligned}I_2 &= 2 - I_0 = 2 - \arctan(2) \approx .89285 \\ I_3 &= 2 - I_1 = 2 - \ln(5)/2 \approx 1.19528 \\ I_4 &= \frac{8}{3} - I_2 = \frac{2}{3} + \arctan(2) \approx 1.77382\end{aligned}$$

3. d. The task here is to find the fourth degree polynomial which best fits $1/(1+x^2)$ over $[0, 2]$. As in the other ex-

ercises, we set $Q(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$. To find the a_n we start with integral equations of the form

$$\int \frac{x^n dx}{1+x^2} = \int (a_0x^n + a_1x^{n+1} + a_2x^{n+2} + a_3x^{n+3} + a_4x^{n+4}) dx$$

The left hand side is I_n , and the right-hand side can be evaluated directly. When that is done we get these five equations:

$$\begin{aligned} \arctan(2) &= 2a_0 + 2a_1 + \frac{8}{3}a_2 + 4a_3 + \frac{32}{5}a_4 \\ \ln(5)/2 &= 2a_0 + \frac{8}{3}a_1 + 4a_2 + \frac{32}{5}a_3 + \frac{32}{3}a_4 \\ 2 - \arctan(2) &= \frac{8}{3}a_0 + 4a_1 + \frac{32}{5}a_2 + \frac{32}{3}a_3 + \frac{128}{7}a_4 \\ 2 - \ln(5)/2 &= 4a_0 + \frac{32}{5}a_1 + \frac{32}{3}a_2 + \frac{128}{7}a_3 + 32a_4 \\ \frac{2}{3} + \arctan(2) &= \frac{32}{5}a_0 + \frac{32}{3}a_1 + \frac{128}{7}a_2 + 32a_3 + \frac{512}{9}a_4 \end{aligned}$$

Mathematica's `Solve` command yields the following:

$$\begin{aligned} a_0 &= 1.00707 & a_1 &= -.0691842 \\ a_2 &= -1.01592 & a_3 &= .732809 \\ a_4 &= -.154802 \end{aligned}$$

Thus,

$$\begin{aligned} Q(x) &= 1.00707 - .0691842x - 1.01592x^2 \\ &\quad + .732809x^3 - .154802x^4. \end{aligned}$$

This differs slightly from the expression found in the text (page 671).

The maximum difference between $1/(1+x^2)$ and $Q(x)$ occurs at the right endpoint, $x = 2$, and has the value $\frac{1}{5} - Q(2) = .0093384$. Since $.0093384 < .5 \times 10^{-1}$, $Q(x)$ and $1/(1+x^2)$ agree to at least one decimal place over the entire interval.

4. Here $Q(x) = a_0 + a_1x + \cdots + a_6x^6$ and the integral equations have the form

$$\int_{-\pi}^{\pi} x^n \sin(x) dx = \int_{-\pi}^{\pi} (a_0x^n + a_1x^{n+1} + \cdots + a_6x^{n+6}) dx.$$

If we let $I_n = \int_{-\pi}^{\pi} x^n \sin(x) dx$, then

$$I_0 = \int_{-\pi}^{\pi} \sin x dx = 0,$$

which is obvious from the shape of the graph of $y = \sin(x)$

over $[-\pi, \pi]$. Next,

$$\begin{aligned} I_1 &= \int_{-\pi}^{\pi} x \sin(x) dx \\ &= -x \cos(x) \Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \cos(x) dx \\ &= -x \cos(x) + \sin(x) \Big|_{-\pi}^{\pi} = 2\pi. \end{aligned}$$

Except for the lower limit of integration, this is the same as in exercise 1. Likewise, we can get a recursion relation for the I_n :

$$\begin{aligned} I_{n+2} &= \int_{-\pi}^{\pi} x^{n+2} \sin(x) dx \\ &= -x^{n+2} \cos(x) + (n+2)x^{n+1} \sin(x) \Big|_{-\pi}^{\pi} \\ &\quad - (n+2)(n+1) \int_{-\pi}^{\pi} x^n \sin(x) dx \\ &= \pi^{n+2} - (-\pi)^{n+2} - (n+2)(n+1)I_n \end{aligned}$$

The value of the last expression depends on n :

$$I_{n+2} = \begin{cases} 2\pi^{n+2} - (n+2)(n+1)I_n & n \text{ odd} \\ 0 - (n+2)(n+1)I_n & n \text{ even} \end{cases}$$

In fact, since $I_0 = 0$, we have $I_2 = 0$, $I_4 = 0$, $I_6 = 0$, and so on. Thus

$$I_{n+2} = \begin{cases} 2\pi^{n+2} - (n+2)(n+1)I_n & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Therefore $I_3 = 2\pi^3 - 6I_1 = 2\pi^3 - 12\pi$, $I_5 = 2\pi^5 - 20I_3 = 2\pi^5 - 40\pi^3 + 240\pi$. The integral equations lead to seven ordinary equations which it is useful to break into one group of four and another group of three:

$$\begin{aligned} 0 &= 2\pi a_0 + \frac{2\pi^3}{3}a_2 + \frac{2\pi^5}{5}a_4 + \frac{2\pi^7}{7}a_6 \\ 0 &= \frac{2\pi^3}{3}a_0 + \frac{2\pi^5}{5}a_2 + \frac{2\pi^7}{7}a_4 + \frac{2\pi^9}{9}a_6 \\ 0 &= \frac{2\pi^5}{5}a_0 + \frac{2\pi^7}{7}a_2 + \frac{2\pi^9}{9}a_4 + \frac{2\pi^{11}}{11}a_6 \\ 0 &= \frac{2\pi^7}{7}a_0 + \frac{2\pi^9}{9}a_2 + \frac{2\pi^{11}}{11}a_4 + \frac{2\pi^{13}}{13}a_6 \end{aligned}$$

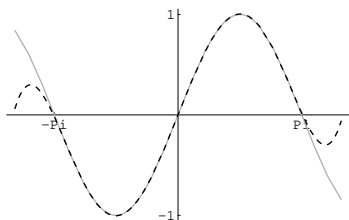
The solution here is $a_0 = a_2 = a_4 = a_6 = 0$.

$$\begin{aligned} 2\pi &= \frac{2\pi^3}{3}a_1 + \frac{2\pi^5}{5}a_3 + \frac{2\pi^7}{7}a_5 \\ 2\pi^3 - 12\pi &= \frac{2\pi^5}{5}a_1 + \frac{2\pi^7}{7}a_3 + \frac{2\pi^9}{9}a_5 \\ 2\pi^5 - 40\pi^3 + 240\pi &= \frac{2\pi^7}{7}a_1 + \frac{2\pi^9}{9}a_3 + \frac{2\pi^{11}}{11}a_5 \end{aligned}$$

Mathematica's `Solve` command yields the following:

$$a_1 = 0.987862, a_3 = -0.155271, a_5 = 0.00564312.$$

Thus $Q(x) = 0.987862x - 0.155271x^3 + 0.00564312x^5$.



The maximum difference between $Q(x)$ and $\sin(x)$ is at the endpoints, and $Q(\pi) - \sin(\pi) = .0159902$. Thus $Q(x)$ agrees with $\sin(x)$ to at least one decimal place of accuracy over the entire interval $[-\pi, \pi]$.

5. As in the text, we measure S and I in units of 10,000 persons. Thus $S(0) = 4.54$, $I(0) = .21$. (Recall that R doesn't appear in the equations for S' and I' .) We also convert time to "decadays", so $0 \leq t \leq 4$. The differential equations take the form on page 673 of the text.

As always, we start with the integral equations

$$\int_0^4 t^n S(t) dt = \int_0^4 (a_0 t^n + a_1 t^{n+1} + \cdots + a_8 t^{n+8}) dt.$$

To calculate the integral of $t^n S(t)$ we then use ideas given in the text. We use a Basic program to get an array of 2000 values of S over $0 \leq t \leq 4$ and then use Simpson's rule to integrate this with t^n . Once that is done, we can write down nine equations in the nine unknowns $a_0, a_1, \dots, a_8$ and then solve them using Mathematica.

The following program solves the differential equations for $S(t)$ and $I(t)$ and stores the values in arrays in the first FOR-NEXT loop. The second FOR-NEXT loop integrates $t^j S(t)$ for each integer j from 0 to 8. The FOR-NEXT loop that is embedded within this loop actually implements Simpson's rule. In the first calculation, $S(0)$ is the left endpoint value, $S(1)$ is the midpoint value, and $S(2)$ is the right endpoint value. In the second calculation, $S(2)$ is the left endpoint value, and so on. The last calculation uses the values $S(1998)$, $S(1999)$, and $S(2000)$.

```
DEFDBL A-Z
DIM S(2000), I(2000)
S(0) = 4.54
I(0) = 0.21
c = 0
d = 4
numberofsteps = 1000
deltat = (d - c) / numberofsteps
al = 0
```

```
am = 0
ar = 0
FOR k = 0 to 2 * numberofsteps - 1
  Sprime = - S(k) * I(k)
  Iprime = S(k) * I(k) - I(k) / 1.4
  deltaS = Sprime * deltat / 2
  deltaI = Iprime * deltat / 2
  S(k + 1) = S(k) + deltaS
  I(k + 1) = I(k) + deltaI
NEXT k
FOR j = 0 to 8
  t = 0
  al = 0
  am = 0
  ar = 0
  FOR l = 0 to 2 * numberofsteps - 2 STEP 2
    deltaAl = t^j * S(l) * deltat
    t = t + deltat / 2
    deltaAm = t^j * S(l + 1) * deltat
    t = t + deltat / 2
    deltaAr = t^j * S(l + 2) * deltat
    al = al + deltaAl
    am = am + deltaAm
    ar = ar + deltaAr
  NEXT l
  simpson = (al + 4 * am + ar) / 6
  PRINT j, simpson
NEXT j [Note: if your computer does not define 0^0 = 1,
you will have to add an additional loop which calculates
the j = 0 case which would omit the t ^ j term from
the definition of deltaAl or add an if clause to take
care of this special case.]
```

The program yields the following:

$$\begin{array}{ll} I_0 = 3.748141 & I_1 = 2.106287 \\ I_2 = 2.094904 & I_3 = 3.128982 \\ I_4 = 6.410054 & I_5 = 16.309857 \\ I_6 = 47.435403 & I_7 = 149.502105 \\ I_8 = 495.298231 \end{array}$$

After we integrate the right-hand sides of the integral equations we get the following:

$$\begin{aligned} 3.748141 &= 4 a_0 + \frac{4^2}{2} a_1 + \cdots + \frac{4^9}{9} a_8 \\ 2.106287 &= \frac{4^2}{2} a_0 + \frac{4^3}{3} a_1 + \cdots + \frac{4^{10}}{10} a_8 \\ 2.094904 &= \frac{4^3}{3} a_0 + \frac{4^4}{4} a_1 + \cdots + \frac{4^{11}}{11} a_8 \\ 3.128982 &= \frac{4^4}{4} a_0 + \frac{4^5}{5} a_1 + \cdots + \frac{4^{12}}{12} a_8 \\ 6.410054 &= \frac{4^5}{5} a_0 + \frac{4^6}{6} a_1 + \cdots + \frac{4^{13}}{13} a_8 \end{aligned}$$

$$\begin{aligned}
16.309857 &= \frac{4^6}{6}a_0 + \frac{4^7}{7}a_1 + \cdots + \frac{4^{14}}{14}a_8 \\
47.435403 &= \frac{4^7}{7}a_0 + \frac{4^8}{8}a_1 + \cdots + \frac{4^{15}}{15}a_8 \\
149.502105 &= \frac{4^8}{8}a_0 + \frac{4^9}{9}a_1 + \cdots + \frac{4^{16}}{16}a_8 \\
495.298231 &= \frac{4^9}{9}a_0 + \frac{4^{10}}{10}a_1 + \cdots + \frac{4^{17}}{17}a_8
\end{aligned}$$

Mathematica's `Solve` command obtains the following:

$$\begin{aligned}
a_0 &= 4.46314 & a_1 &= .331874 \\
a_2 &= -5.35026 & a_3 &= -2.62142 \\
a_4 &= 9.2653 & a_5 &= -6.56572t^5 \\
a_6 &= 2.15279 & a_7 &= -0.34489 \\
a_8 &= .0218492
\end{aligned}$$

Thus the best-approximating polynomial is

$$\begin{aligned}
Q(t) &= 4.46314 + .331874t - 5.35026t^2 \\
&\quad - 2.62142t^3 + 9.2653t^4 - 6.56572t^5 \\
&\quad + 2.15279t^6 - 0.34489t^7 + .0218492t^8.
\end{aligned}$$

If we double the size of the arrays (by declaring `DIM S(4000), I(4000)` at the start of the program), we get different values for I_n and hence a_n :

$$\begin{aligned}
I_0 &= 3.748141 & I_1 &= 2.106040 \\
I_2 &= 2.096918 & I_3 &= 3.136582 \\
I_4 &= 6.433915 & I_5 &= 16.384547 \\
I_6 &= 47.675976 & I_7 &= 150.301063 \\
I_8 &= 498.021777
\end{aligned}$$

(Notice the greatest differences are in the higher I_n .) The best-approximating polynomial is now

$$\begin{aligned}
Q(t) &= 4.463 + .338025t - 5.41663t^2 \\
&\quad - 2.48697t^3 + 9.14573t^4 - 6.50925t^5 \\
&\quad + 2.13799t^6 - 0.342855t^7 + .0217342t^8.
\end{aligned}$$

Solutions: Chapter 11

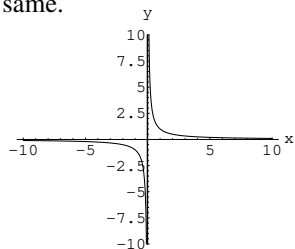
Techniques of Integration

11.1 Antiderivatives

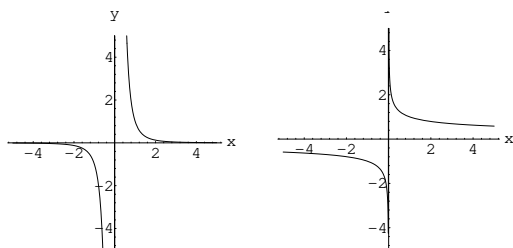
We'll use the following conventions: The inverse of $y = f(x)$ will be found by "solving for x ": $x = g(y)$. These two relations between x and y have exactly the same graph. However, if we switch the variables in $x = g(y)$ and write $y = g(x)$, this is equivalent to reflecting the x, y -plane across the line $y = x$ (interchanging the x and y axes). The graph of $y = g(x)$ is therefore the reflection of the graph of $y = f(x)$ across the line $y = x$.

• Also, we'll skip the "+C" most of the time.

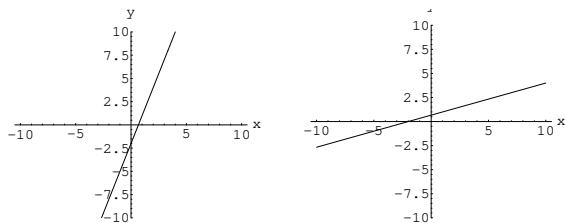
1. The inverse of $y = 1/x$ is $x = 1/y$, so $g(x) = 1/x$. The graphs are the same.



2. a. The inverse of $y = 1/x^3$ is $g(x) = 1/\sqrt[3]{x}$.



2. b. The inverse of $y = 3x - 2$ is $g(x) = (x + 2)/3$.



3. a. The picture the text refers to appears on page 686. In this picture we see $\tan(\theta) = x$. If we look at the complementary angle $\frac{\pi}{2} - \theta$ at the top of the triangle, we can see $\tan(\frac{\pi}{2} - \theta) = 1/x$. Therefore $\arctan(1/x) = \frac{\pi}{2} - \theta$, so

$$\arctan(x) + \arctan(1/x) = \theta + \frac{\pi}{2} - \theta = \frac{\pi}{2}.$$

3. b. $(\arctan(x))' + (\arctan(1/x))' = (\pi/2)' = 0$, so $(\arctan(1/x))' = -1/(1+x^2)$.

3. c. Let $u = 1/x$; then $y = \arctan(1/x) = \arctan(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = \frac{1}{1+u^2} \frac{-1}{x^2} = -\frac{1}{x^2+x^2u^2} = -\frac{1}{x^2+1},$$

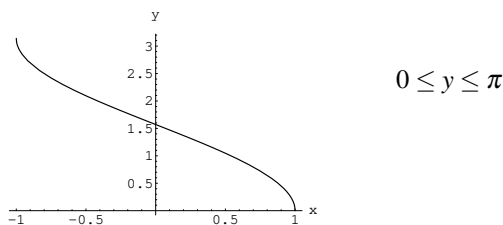
because $xu = 1$.

4. The key is that dy/dx is the reciprocal of dx/dy , and $b^y = x$:

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{\ln b \cdot b^y} = \frac{1}{\ln b \cdot x} = \frac{1}{\ln(b)} \cdot \frac{1}{x}.$$

5. a. Since $y = \cos(x)$ is one-to-one when $0 \leq x \leq \pi$ (and $-1 \leq y \leq 1$), we define the inverse function $\arccos(x)$ for $-1 \leq x \leq 1$.

5. b.



5. c. Let $x = \cos(y)$, then $y = \arccos(x)$ and

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{-\sin(y)} = \frac{-1}{\sqrt{1-\cos^2(y)}} = \frac{-1}{\sqrt{1-x^2}}.$$

6. a. The picture is at the top of page 685. In the picture we see $\sin(\theta) = x$. If we look at the complementary angle $\frac{\pi}{2} - \theta$ at the top of the triangle, we can see $\cos(\frac{\pi}{2} - \theta) = x$. Therefore $\arcsin(x) = \frac{\pi}{2} - \theta$, so

$$\arcsin(x) + \arccos(x) = \theta + \frac{\pi}{2} - \theta = \frac{\pi}{2}.$$

6. b. $(\arcsin(x))' + (\arccos(x))' = (\pi/2)' = 0$, so $(\arccos(x))' = -(\arcsin(x))' = -1/\sqrt{1-x^2}$. This agrees with what we found in the preceding exercise.

7. $F(x) = \int dx/\sqrt{1-x^2}$ is an antiderivative of $1/\sqrt{1-x^2}$, but $(\arcsin(x))' = 1/\sqrt{1-x^2}$ so we can write

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C.$$

8. This just involves differentiating each antiderivative, using the differentiation rules, and seeing that it equals the given function.

9. We don't bother to add the "+ C" because the exercise asks only for an antiderivative.

$$\begin{array}{ccc} 3t & \frac{5t^2}{2} & -\frac{5t^2}{2} \\ 3t - \frac{5t^2}{2} & \frac{7x^5}{5} & -\frac{1}{2y^2} \\ \frac{e^{2z}}{2} & \frac{u^2}{2} + \ln u & \frac{w^7}{7} + \frac{w^4}{2} + w \\ \frac{1}{5} \sin(5v) & \frac{x^{10}}{10} + \frac{5x^8}{8} - \frac{x^6}{3} & \frac{\sin^2 t}{2} \end{array}$$

10. a. $\int 3x dx = \frac{3}{2}x^2.$

10. b. $\int e^z dz = e^z.$

10. c. $\int 7y + \frac{1}{y} dy = \frac{7}{2}y^2 + \ln y.$

10. d. $\int 5 \sin(w) - 2 \cos(w) dw = -5 \cos(w) - 2 \sin(w).$

10. e. $\int e^{z+2} dz = e^{z+2}.$

10. f. $\int \frac{5}{1+r^2} dr = 5 \arctan(r).$

10. g. $\int 3u du = \frac{3}{2}u^2.$

10. h. $\int 5t^4 dt = t^5.$

10. i. $\int 7y - \frac{4}{y^2} dy = \frac{7}{2}y^2 + \frac{4}{y}.$

10. j. $\int dx = x.$

10. k. $\int \cos(4x) dx = \frac{1}{4} \sin(4x).$

10. l. $\int \frac{1}{\sqrt{1-4s^2}} ds = \frac{1}{2} \arcsin 2s.$

11. a. $F(x) = 7x + C$; therefore $F(0) = C = 12$ and $F(x) = 7x + 12$.

11. b. $G(x) = 7x + C$ and $G(3) = 7 \cdot 3 + C = 1$ so $C = -20$ and $G(x) = 7x - 20$.

11. c. $F(x) - G(x) = 32$ for all x .

12. a. $F(t) = \frac{1}{2}t^2 + \sin(t) + C$; therefore $F(0) = C = 3$ so $F(t) = \frac{1}{2}t^2 + \sin(t) + 3$.

12. b. $G(t) = \frac{1}{2}t^2 + \sin(t) + C$; therefore $G(\pi/2) = \pi^2/8 + 1 + C = -5$ so $C = -6 - \pi^2/8$ and $G(t) = \frac{1}{2}t^2 + \sin(t) - 6 - \frac{1}{8}\pi^2$.

12. c. $F(t) - G(t) = 9 + \frac{1}{8}\pi^2$ for all x .

13. $ay + \frac{1}{2}by^2.$

14. a. $((1+x^3)^{10})' = 10(1+x^3)^9 \cdot 3x^2 = 30x^2(1+x^3)^9.$

14. b. $\frac{1}{30}(1+x^3)^{10}.$

14. c. $\frac{1}{3}x^3 + \frac{1}{30}(1+x^3)^{10}.$

15. a. $(x \ln x)' = x \cdot (1/x) + \ln x \cdot 1 = 1 + \ln x.$

15. b. Since $\ln(x) = (1 + \ln(x)) - 1$, we can get an antiderivative for $\ln(x)$ by adding together an antiderivative for $1 + \ln(x)$ (determined in part a) and an antiderivative for -1 (we can use $-x$). Thus the antiderivative we get is $x \ln(x) - x$.

16. a. $G'(y) = (\ln(2y))' = (1/2y) \cdot 2 = (1/y).$

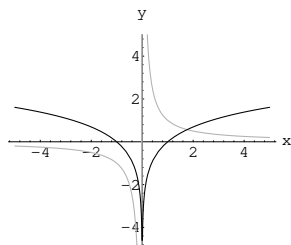
16. b. If $\ln(2y) = \ln(y) + C$ for some C , then when $y = 1$ we have $\ln(2) = \ln(1) + C = 0 + C = C$. Thus $C = \ln(2)$.

Alternately, we can use $\ln(ab) = \ln(a) + \ln(b)$: $\ln(2y) = \ln(y) + \ln(2)$, so $C = \ln(2)$.

17. First, $(-\cos^2(t))' = -2\cos(t)(-\sin(t)) = 2\cos(t)\sin(t)$, so $\cos^2(t)$ is an antiderivative. We know $\sin^2(x) + \cos^2(x) = 1$ for all x , so $-\cos^2(t) = \sin^2(t) - 1$. Therefore $C = -1$.

18. a. By the chain rule, $(\ln(-x))' = -(1/x) \cdot -1 = (1/x)$, so $\ln(-x)$ is an antiderivative of $1/x$.

18. b.



$F(x)$: black;
 $1/x$: gray.

Note that, when $x < 0$, the slope of $F(x)$ is negative and becomes more and more negative as $x \rightarrow 0$. This agrees in a qualitative way with the values of $y = 1/x$ when $x < 0$.

18. c. By definition,

$$|x| = \begin{cases} -x & x \leq 0 \\ x & x \geq 0 \end{cases}$$

Now refer to the way $F(x)$ is defined. It uses the same cases as $|x|$, and we see $F(x) = \ln|x|$.

18. d. $G'(x) = 1/x$, just as was true for $F(x)$. The left half of the graph of G is the same as the left half of F , but the right half is raised one unit in comparison to the right half of F . There is no C for which $\ln|x| + C = G(x)$ for all x , because every graph $y = \ln|x| + C$ is symmetric about the y -axis, but G lacks this symmetry (its right side is higher than its left).

18. e. The general form of the antiderivative of $1/x$ is

$$H(x) = \begin{cases} \ln(-x) + C_1 & x < 0 \\ \ln(x) + C_2 & x > 0 \end{cases}$$

The left half of the graph of H matches the left half of F shifted up by C_1 ; the right half of H matches the right half of F shifted up by C_2 .

19. a. The domain of the $\tan(x)$ breaks down into infinitely many separate intervals: $(-\frac{\pi}{2} + n\pi, \frac{\pi}{2} + n\pi)$, for every integer n .

19. b. For each integer n , $\cos(x) = 0$ when $x = \frac{\pi}{2} + 2n\pi$, while $\cos(x) < 0$ when $\frac{1}{2}\pi + 2n\pi < x < \frac{3}{2}\pi + 2n\pi$.

19. c. When $\cos(x) < 0$, that is, when $\frac{1}{2}\pi + 2n\pi < x < \frac{3}{2}\pi + 2n\pi$ (for any integer n), let

$$\int \tan(x) dx = -\ln(-\cos(x)).$$

19. d. For example, the function

$$F(x) = \begin{cases} -\ln(\cos(x)) + 1 & \cos(x) > 0 \\ -\ln(-\cos(x)) & \cos(x) < 0 \end{cases}$$

is an antiderivative of $\tan(x)$ but is *not* of the form $-\ln|\cos(x)| + C$ for any C whatsoever.

19. e. Here is one way to proceed: For each integer n , choose a constant C_n . Then define

$$\int \tan(x) dx = -\ln|\cos(x)| + C_n$$

for x in the interval $\frac{1}{2}\pi + n\pi < x < \frac{3}{2}\pi + n\pi$.

19. f. There are many possibilities. Here is one:

$$G(x) = \begin{cases} -\ln|\cos x| + 5 & x < \pi/2 \\ -\ln|\cos x| - 23 & \pi/2 < x < 33\pi/2 \\ -\ln|\cos x| + 197 & 33\pi/2 < x \end{cases}$$

20. The domain of $x^{p+1}/p+1$ will consist of two separate pieces, namely $x < 0$ and $x > 0$, when $p+1 < 0$, because then x^{p+1} is undefined at $x = 0$.

21. a. $F(x) = x^3/3 + x^4/4$, so $F'(x) = x^2 + x^3$. The fundamental theorem also gives this directly.

21. b. $F(x) = \ln(x)$, so $F'(x) = 1/x$. The fundamental theorem also gives this directly.

21. c. By the fundamental theorem, $F'(x) = x/(1+x^3)$. (We don't try to get an explicit formula for $F(x)$ itself.)

21. d. $F(x) = \sin(x^2)$, so $F'(x) = 2x\cos x^2$. The fundamental theorem also gives this directly (cf. part e).

21. e. Let $G(u) = \int_1^u \frac{v}{1+v^3} dv$. Then $F(x) = G(x^2)$, so $F'(x) = G'(x^2) \cdot 2x$ by the chain rule. The fundamental theorem tells us $G'(u) = u/(1+u^3)$, so we finally get $F'(x) = 2x \cdot x^2/(1+x^6) = 2x^3/(1+x^6)$.

22. a. By the chain rule, $F'(x) = \frac{1}{2}(1+x^2)^{-1/2} \cdot 2x = x/\sqrt{1+x^2}$ and $F(0) = \sqrt{1+0^2} - 1 = 1 - 1 = 0$.

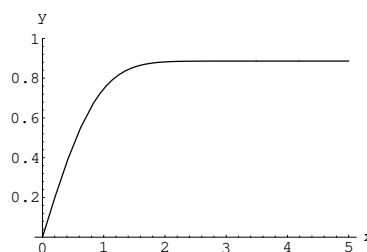
By the fundamental theorem, $G'(x) = x/\sqrt{1+x^2}$ and $G(0) = 0$.

22. b. The given integrals are $G(1)$, $G(2)$, and $G(5)$. Since $G(x) = F(x)$ for every x , we evaluate the various integrals (i.e., the various $G(x)$) by evaluating $F(x)$.

$$F(1) = \sqrt{2} - 1, \quad F(2) = \sqrt{5} - 1, \quad F(5) = \sqrt{26} - 1.$$

23. a. The output of this part is the graph in part b.

23. b.



23. c. $e^{-x^2} > 0$ for all x , so the slope of $A(x)$ is always positive, and its graph will increase from left to right as x increases.

23. d. $A(5) \approx .886226925451$, $A(10) \approx .886226925453$, $A(30) \approx .886226925453$.

23. e. The graph of $A(x)$ appears to level off well before $x = 30$, so $\lim_{x \rightarrow \infty} A(x) \approx .886226925453$. (More subtle methods show that the limit is $\sqrt{\pi}/2 = .8862269254527\dots$. A good numerical integrator should get $A(30)$ very close to this value.)

23. f. The integrals are $A(1) \approx .7468241$ and $A(2) \approx .8820814$.

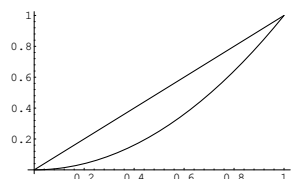
23. g. This is $A(2) - A(1) \approx .1352573$.

24. $\int_1^4 x^3 + x \, dx = 71.25$.

25. $\int_0^{\ln(3)} e^{3x} \, dx = \frac{1}{3} e^{3x} \Big|_0^{\ln(3)} = \frac{1}{3} (e^{3\ln(3)} - e^0) = \frac{1}{3} (3^3 - 1) = 26/3$.

26. a. The average values are $1/2$, $1/3$, $1/4$, and $1/5$, respectively, for $y = x$, x^2 , x^3 , and x^4 .

26. b. For every x in the open interval $0 < x < 1$, $x^2 \leq x$, so we would expect the average value of x^2 to be less than the average value of x as well.



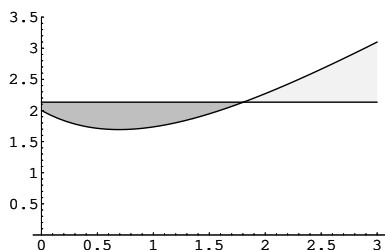
27. a. From the graph we see that the maximum of $f(x) = x + 2e^{-x}$ is attained at the endpoint $x = 3$, and $f(3) \approx 3.099574$.

The minimum occurs at an interior critical point; we find this by solving $f'(x) = 1 - 2e^{-x} = 0$. Thus $e^x = 2$ so $x = \ln(2)$ and $f(x) = \ln(2) + 2e^{-\ln(2)} = \ln(2) + 1 \approx 1.693147$.

The average is found from

$$\mu = \frac{1}{3} \int_0^3 x + 2e^{-x} \, dx = \frac{x^2}{6} - \frac{2e^{-x}}{3} \Big|_0^3 \approx 2.1334753.$$

27. b.



27. c. $\mu \approx 2.1334753$, and the graph of $f(x)$ lies above the line $y = \mu$ when $x > 1.8043$, approximately. The graph of $f(x)$ lies below $y = \mu$ when $0 \leq x < 1.8043$. The area above f and below $y = \mu$ has the darker shading; the area below f and above $y = \mu$ has the lighter shading. When μ is the average value of f these two regions should have the same area—and that is what appears to be the case.

11.2 Integration by Substitution

1. All antiderivatives are given without the added constant “+C” and simple forms are used rather than more elaborate ones—e.g., $\ln(x)$ rather than $\ln|x|$.

1. a. Method 1: $f(u) = u^{50}$, $F(u) = u^{51}/51$, $G(y) = y^2 + 1$.

$$\int 2y(y^2 + 1)^{50} \, dy = F(G(y)) = (y^2 + 1)^{51}/51.$$

Method 2: $u = y^2 + 1$, $du = 2y \, dy$.

$$\int 2y(y^2 + 1)^{50} \, dy = \int u^{50} \, du = u^{51}/51 = (y^2 + 1)^{51}/51.$$

1. b. Method 1: $f(u) = \frac{1}{5} \sin(u)$, $F(u) = -\frac{1}{5} \cos(u)$, $G(z) = 5z$, $g(z) = 5$.

$$\int \sin(5z) \, dz = F(G(z)) = -\frac{1}{5} \cos(5z).$$

Method 2: $u = 5z$, $du = 5 \, dz$.

$$\int \sin(5z) \, dz = \frac{1}{5} \int \sin(u) \, du = -\frac{1}{5} \cos(u) = -\frac{1}{5} \cos(5z).$$

1. c. Method 1: $f(u) = 2e^u$, $F(u) = 2e^u$, $G(x) = \sqrt{x}$, $g(x) = 1/2\sqrt{x}$.

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx = F(G(x)) = 2e^{\sqrt{x}}.$$

Method 2: $u = \sqrt{x}$, $du = dx/2\sqrt{x}$.

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx = 2 \int e^u \, du = 2e^u = 2e^{\sqrt{x}}.$$

1. d. Method 1: $f(u) = \frac{1}{5} u^{50}$, $F(u) = \frac{1}{5 \cdot 51} u^{51}$, $G(t) = 5t + 7$, $g(t) = 5$.

$$\int (5t + 7)^{50} \, dt = F(G(t)) = \frac{1}{5 \cdot 51} (5t + 7)^{51}.$$

Method 2: $u = 5t + 5$, $du = 5 dt$.

$$\int (5t + 7)^{50} dt = \frac{1}{5} \int u^{50} du = \frac{u^{51}}{5 \cdot 51} = \frac{(5t + 7)^{51}}{5 \cdot 51}.$$

1. e. Method 1: $f(w) = \sqrt[3]{w}$, $F(w) = \frac{3}{4}w^{4/3}$, $G(u) = u^3 + 8$, $g(u) = 3u^2$.

$$\int 3u^2 \sqrt[3]{u^3 + 8} du = F(G(u)) = \frac{3}{4}(u^3 + 8)^{4/3}.$$

Method 2: $w = u^3 + 8$, $dw = 3u^2 du$.

$$\int 3u^2 \sqrt[3]{u^3 + 8} du = \int \sqrt[3]{w} dw = \frac{3}{4}w^{4/3} = \frac{3}{4}(u^3 + 8)^{4/3}.$$

1. f. $u = 2v + 1$, $du = 2 dv$.

$$\int \frac{1}{2v + 1} dv = \frac{1}{2} \int du/u = \frac{1}{2} \ln(u) = \frac{1}{2} \ln(2v + 1).$$

1. g. $\tan(x) = \sin(x)/\cos(x)$; $u = \cos(x)$, $du = -\sin(x) dx$.

$$\int \tan(x) dx = - \int du/u = -\ln(u) = -\ln(\cos(x)).$$

1. h. $u = \tan(x)$, $du = \sec^2(x) dx$.

$$\int \tan^2(x) \sec^2(x) dx = \int u^2 du = \frac{1}{3}u^3 = \frac{1}{3} \tan^3(x).$$

1. i. $\sec(x/2) \tan(x/2) = \sin(x/2)/\cos^2(x/2)$; $u = \cos(x/2)$, $du = -\frac{1}{2} \sin(x/2) dx$.

$$\begin{aligned} \int \sec(x/2) \tan(x/2) dx \\ = -2 \int \frac{du}{u^2} = +\frac{2}{u} = \frac{2}{\cos(x/2)} = 2 \sec(x/2). \end{aligned}$$

1. j. $u = \cos(w)$, $du = -\sin(w) dw$.

$$\begin{aligned} \int \sin(w) \sqrt{\cos(w)} dw \\ = - \int \sqrt{u} du = -\frac{2}{3}u^{3/2} = -\frac{2}{3} \cos^{3/2}(w). \end{aligned}$$

1. k. $u = \sqrt{s}$, $du = 1/2 \sqrt{s} ds$.

$$\begin{aligned} \int \frac{\sin(\sqrt{s})}{\sqrt{s}} ds \\ = 2 \int \sin(u) du = -2 \cos(u) = -2 \cos(\sqrt{s}). \end{aligned}$$

1. l. $u = 3 - x$, $du = -dx$.

$$\int \sqrt{3 - x} dx = - \int \sqrt{u} du = -\frac{2}{3}u^{3/2} = -\frac{2}{3}(3 - x)^{3/2}.$$

1. m. $u = \ln(r)$, $du = dr/r$.

$$\int \frac{dr}{r \ln(r)} = \int du/u = \ln(u) = \ln(\ln(r)).$$

1. n. $u = 1 + e^x$, $du = e^x dx$.

$$\begin{aligned} \int e^x \sin(1 + e^x) dx &= \int \sin(u) du \\ &= -\cos(u) = -\cos(1 + e^x). \end{aligned}$$

1. o. $u = 1 + y^2$, $du = 2y dy$

$$\int \frac{y}{1 + y^2} dy = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln u = \frac{1}{2} \ln(1 + y^2).$$

1. p. $u = 1 - w^2$, $du = -2w dw$

$$\int \frac{w}{\sqrt{1 - w^2}} dw = -\frac{1}{2} \int \frac{du}{\sqrt{u}} = -1 \sqrt{u} = -\sqrt{1 - w^2}.$$

1. q. $u = 2y$, $du = 2 dy$.

$$\begin{aligned} \int \frac{1}{1 + 4y^2} dy &= \frac{1}{2} \int \frac{1}{1 + u^2} du \\ &= \frac{1}{2} \arctan(u) = \frac{1}{2} \arctan(2y). \end{aligned}$$

1. r. $u = 3w$, $du = 3 dw$

$$\begin{aligned} \int \frac{1}{\sqrt{1 - 9w^2}} dw &= \frac{1}{3} \int \frac{du}{\sqrt{1 - u^2}} \\ &= \frac{1}{3} \arcsin(u) = \frac{1}{3} \arcsin(3w). \end{aligned}$$

2. a. Method 1: An antiderivative of $e^s/(1 + e^s)$ is $\ln(1 + e^s)$; therefore

$$\int_0^1 \frac{e^s}{1 + e^s} ds = \ln(1 + e^s) \Big|_0^1 = \ln(1 + e) - \ln(2).$$

Method 2: Let $u = 1 + e^s$, $du = e^s ds$; then $u = 2$ when $s = 0$ and $u = 1 + e$ when $s = 1$, so

$$\int_0^1 \frac{e^s}{1 + e^s} ds = \int_2^{1+e} \frac{du}{u} = \ln u \Big|_2^{1+e} = \ln(1 + e) - \ln(2).$$

2. b. $\ln(e) = 1$, so this is the same as part a.

2. c. Method 2 (only): $u = 2x + 1$, $du = 2dx$, then $u = 3$ when $x = 1$ and $u = 7$ when $x = 3$, so

$$\int_1^3 \frac{1}{2x+1} dx = \frac{1}{2} \int_3^7 \frac{du}{u} = \frac{1}{2} \ln u \Big|_3^7 = \frac{\ln(7) - \ln(3)}{2}.$$

2. d. If we let $u = -(2x + 1)$, $du = -2dx$, then $u = 5$ when $x = -3$ and $u = 1$ when $x = -1$, so

$$\int_{-3}^{-1} \frac{1}{2x+1} dx = \frac{1}{2} \int_5^1 \frac{du}{u} = \frac{1}{2} \ln u \Big|_5^1 = -\frac{\ln(5)}{2}.$$

2. e. Let $u = 1 + t^2$, $du = 2t dt$. Then $u = 1$ when $t = 0$ and $u = 2$ when $t = 1$, so

$$\int_0^1 \frac{t}{\sqrt{1+t^2}} dt = \frac{1}{2} \int_1^2 \frac{du}{\sqrt{u}} = \sqrt{u} \Big|_1^2 = \sqrt{2} - 1.$$

2. f. Let $u = \pi\sqrt{t}$, $du = \pi dt/2\sqrt{t}$. Then $u = 0$ when $t = 0$ and $u = \pi$ when $t = 1$, so

$$\begin{aligned} \int_0^1 \frac{\sin(\pi\sqrt{t})}{\sqrt{t}} dt &= \frac{2}{\pi} \int_0^\pi \sin(u) du \\ &= -\frac{2}{\pi} \cos(u) \Big|_0^\pi = -\frac{2}{\pi} (-1 - 1) = \frac{4}{\pi}. \end{aligned}$$

2. g. Let $u = x/2$, $du = dx/2$; then $u = 0$ when $x = 0$ and $u = 1$ when $x = 2$, so

$$\begin{aligned} \int_0^2 \frac{dx}{1+x^2/4} &= 2 \int_0^1 \frac{du}{1+u^2} \\ &= 2 \arctan(u) \Big|_0^1 = 2 \arctan(1) = \frac{\pi}{2}. \end{aligned}$$

2. h. Let $u = 3y$, $du = 3dy$; then $u = 0$ when $y = 0$ and $u = 1$ when $y = 1/3$, so

$$\begin{aligned} \int_0^{1/3} \frac{dy}{\sqrt{1-9y^2}} &= \frac{1}{3} \int_0^1 \frac{du}{\sqrt{1-u^2}} \\ &= \frac{1}{3} \arcsin(u) \Big|_0^1 = \frac{\arcsin(1) - \arcsin(0)}{3} = \frac{\pi}{6}. \end{aligned}$$

3. a. If $u = \sin(x)$, $du = \cos(x)$, then

$$I = \int u du = \frac{u^2}{2} = \frac{\sin^2(x)}{2}.$$

3. b. If $u = \cos(x)$, $du = -\sin(x) dx$, then

$$I = -\int u du = -\frac{u^2}{2} = \frac{-\cos^2(x)}{2}.$$

3. c. Answers in parts a and b certainly *look* different. However, antiderivatives can differ, but they must differ by at most a constant. That is true here:

$$\frac{\sin^2(x)}{2} - \left(\frac{-\cos^2(x)}{2} \right) = \frac{\sin^2(x) + \cos^2(x)}{2} = \frac{1}{2}$$

$$\mathbf{3. d.} \quad \int_0^{\pi/2} \sin(x) \cos(x) dx = \frac{\sin^2(x)}{2} \Big|_0^{\pi/2} = \frac{1}{2}.$$

$$\int_0^{\pi/2} \sin(x) \cos(x) dx = \frac{-\cos^2(x)}{2} \Big|_0^{\pi/2} = \frac{-0 + 1}{2} = \frac{1}{2}.$$

The two values agree, as we expect.

4. a. Solutions are the antiderivatives

$$F(x) = \int x^2(1+x^3)^{13} dx = \frac{(1+x^3)^{14}}{3 \cdot 14} + C.$$

4. b. If $F(0) = \frac{1}{42}(1+0^3)^{14} + C = \frac{1}{42} + C = 4$, then $C = 4 - \frac{1}{42} = 3\frac{41}{42}$. Therefore,

$$F(x) = \frac{1}{42}(1+x^3)^{14} + 3\frac{41}{42}.$$

4. c. If $F(-1) = 0 + C = 4$, then $C = 4$ so

$$F(x) = \frac{1}{42}(1+x^3)^{14} + 4.$$

5. The solution $y = G(t)$ is an antiderivative of te^{-t^2} , so

$$G(t) = \int te^{-t^2} dt = \int \frac{e^u}{2} du = \frac{e^{-t^2}}{2} + C.$$

Since $G(0) = 3$, $C = 5/2$.

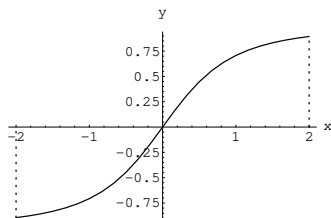
6. a. The average value of $f(x) = x/\sqrt{1+x^2}$ is the integral

$$\begin{aligned} \bar{f} &= \frac{1}{2} \int_0^2 \frac{x}{\sqrt{1+x^2}} dx = \frac{1}{2 \cdot 2} \int_1^5 \frac{du}{\sqrt{u}} \quad (u = 1+x^2) \\ &= \frac{1}{2} \sqrt{u} \Big|_1^5 = \frac{\sqrt{5}-1}{2}. \end{aligned}$$

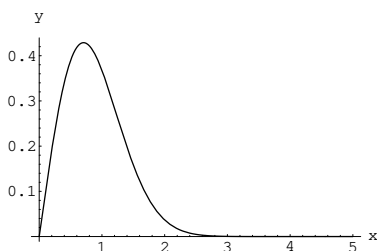
6. b. Notice that the substitution $u = G(x) = 1+x^2$ has the property $G(-2) = G(2) = 5$. In fact, the method on page 705 of the text is valid only when G is either everywhere increasing or everywhere decreasing on $[a, b]$. However, we can still use $u = 1+x^2$ to find the needed antiderivative:

$$\bar{f} = \frac{1}{4} \int_{-2}^2 \frac{xdx}{\sqrt{1+x^2}} = \frac{1}{4} \sqrt{1+x^2} \Big|_{-2}^2 = \frac{\sqrt{5}}{4} - \frac{\sqrt{5}}{4} = 0.$$

Notice that $f(x) = x/\sqrt{1+x^2}$ satisfies the property $f(-x) = -f(x)$ for any x . It is that property that makes the average equal 0 on any interval symmetric about the origin. It says that at points x and $-x$ symmetrically placed around the origin, the value of f are opposites of one another. This property is reflected geometrically in the rotational symmetry of the graph of $f(x)$ around the origin:



7. a.



7. b. Refer to exercise 5 for the antiderivative. The area under the graph of $y = xe^{-x^2}$ from $x = 0$ to $x = 5$ is

$$\int_0^5 xe^{-x^2} dx = -\frac{1}{2}e^{-x^2} \Big|_0^5 = \frac{1}{2} - \frac{1}{2e^{25}}.$$

7. c. The area $A(b)$ under the graph and over the interval $0 \leq x \leq b$ is

$$A(b) = \int_0^b xe^{-x^2} dx = \frac{1}{2} - \frac{1}{2e^{b^2}}.$$

$A(5)$ is indeed the number we found in part a. Furthermore,

$$\begin{aligned} A(10) &= \frac{1}{2} - \frac{1}{2e^{100}} \\ A(100) &= \frac{1}{2} - \frac{1}{2e^{10,000}} \\ A(1000) &= \frac{1}{2} - \frac{1}{2e^{1,000,000}} \end{aligned}$$

Here is another way to write these values that makes it clear that $A(b) \rightarrow .5$ as $b \rightarrow \infty$:

$$\begin{aligned} A(10) &= .5 \text{ to 43 decimal places} \\ A(100) &= .5 \text{ to 4,343 decimal places} \\ A(1000) &= .5 \text{ to 434,294 decimal places} \end{aligned}$$

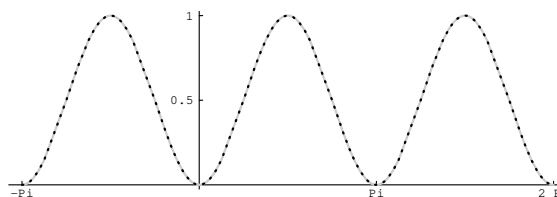
7. d. First, we can say the area between the graph and the entire positive x -axis ought to be the limit of the area $A(b)$ between the graph and the portion of the x -axis given by $0 \leq x \leq b$, as $b \rightarrow \infty$.

Next, we can say that

$$\lim_{b \rightarrow \infty} A(b) = \frac{1}{2} - \lim_{b \rightarrow \infty} \frac{1}{2e^{b^2}}.$$

Since the last limit is 0, we can finally say that total area is exactly $1/2$.

8. a. The graph of $\sin^2(x)$ and $(1 - \cos(2x))/2$ coincide. In the figure below, one of them is shown in solid gray, the other dotted.



8. b. We have $\int \sin^2(x) dx =$

$$\begin{aligned} \int \frac{1 - \cos(2x)}{2} dx &= \int \frac{1}{2} dx - \int \frac{\cos(2x)}{2} dx \\ &= \int \frac{1}{2} dx - \int \frac{\cos(u)}{4} du \\ &= \frac{x}{2} - \frac{\sin(u)}{4} = \frac{x}{2} - \frac{\sin(2x)}{4}. \end{aligned}$$

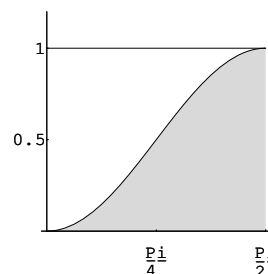
8. c. The average over $[0, \pi]$ is

$$\begin{aligned} \frac{1}{\pi} \int_0^\pi \sin^2(x) dx &= \frac{1}{\pi} \left(\frac{x}{2} - \frac{\sin(2x)}{4} \right) \Big|_0^\pi \\ &= \frac{1}{\pi} \left(\frac{\pi}{2} - \frac{0}{4} \right) = \frac{1}{2}. \end{aligned}$$

The average over $[0, k\pi]$ is the same:

$$\begin{aligned} \frac{1}{k\pi} \int_0^{k\pi} \sin^2(x) dx &= \frac{1}{k\pi} \left(\frac{x}{2} - \frac{\sin(2x)}{4} \right) \Big|_0^{k\pi} \\ &= \frac{1}{k\pi} \left(\frac{k\pi}{2} - \frac{0}{4} \right) = \frac{1}{2}. \end{aligned}$$

8. d.



The graph of $y = \sin^2(x)$ in the interval $[0, \pi/2]$ occupies exactly half of the rectangle $0 \leq y \leq 1$ over the same interval. The symmetry of the graph guarantees this. Consequently, the area under the graph of $y = \sin^2(x)$ is exactly half the area of the rectangle. Therefore the average value of $\sin^2(x)$ is half the average value of the function $y = 1$, or just $1/2$.

An exactly similar conclusion holds for *any* interval of the form $[j\pi/2, n\pi/2]$, for any integers $j < n$. In particular, we can take $j = 0, n = 2k$.

8. e. Both functions satisfy the initial conditions: $f(0) = \sin^2(0) = 0$; $f'(0) = 2\sin(0)\cos(0) = 0$. $g(0) = (1 - \cos(0))/2 = (1 - 1)/2 = 0$; $g'(0) = \sin(2 \cdot 0) = 0$.

Next, $f''(x) = 2\cos^2(x) - 2\sin^2(x)$ while

$$\begin{aligned} 2 - 4f(x) &= 2 - 4\sin^2(x) \\ &= 2\sin^2(x) + 2\cos^2(x) - 4\sin^2(x) \\ &= 2\cos^2(x) - 2\sin^2(x), \end{aligned}$$

so f satisfies the given initial value problem.

Finally,

$$g''(x) = 2\cos(2x)$$

while

$$2 - 4g(x) = 2 - 2(1 - \cos(2x)) = 2\cos(2x)$$

so g satisfies the given initial value problem, too.

But an initial value problem has a unique solution, so we must have $f = \sin^2(x) = (1 - \cos(2x))/2 = g$.

11.3 Integration by Parts

1. a. $F(x) = x, g(x) = \sin(x)$;

$$\int x \sin(x) dx = -x \cos(x) + \sin(x).$$

1. b. $F(t) = t, g(t) = e^t$; $\int t e^t dt = t e^t - e^t$.

1. c. $F(w) = w, g(w) = e^{-w}$;

$$\int w e^{-w} dw = -w e^{-w} - e^{-w}.$$

1. d. $F(x) = \ln(x), g(x) = x$;

$$\int x \ln(x) dx = \frac{x^2 \ln(x)}{2} - \frac{x^2}{4} = \frac{x^2(2\ln(x) - 1)}{4}.$$

1. e. $F(x) = \arcsin(x), g(x) = 1$;

$$\begin{aligned} \int \arcsin(x) dx &= x \arcsin(x) - \int \frac{x}{\sqrt{1-x^2}} dx \\ &= x \arcsin(x) + \sqrt{1-x^2}. \end{aligned}$$

1. f. $F(x) = \arctan(x), g(x) = 1$;

$$\begin{aligned} \int \arctan(x) dx &= x \arctan(x) - \int \frac{x}{1+x^2} dx \\ &= x \arctan(x) - \frac{1}{2} \ln(1+x^2). \end{aligned}$$

1. g. $F(x) = x^2, g(x) = e^{-x}$;

$$\begin{aligned} \int x^2 e^{-x} dx &= -x^2 e^{-x} + 2 \int x e^{-x} dx \\ &= -x^2 e^{-x} - 2x e^{-x} - 2e^{-x}. \end{aligned}$$

The integral on the right hand side is essentially the same as the one given in part c, so we used the solution to part c to complete this problem.

1. h. $F(u) = u^2, g(u) = \cos(u)$;

$$\begin{aligned} \int u^2 \cos(u) du &= u^2 \sin(u) - 2 \int u \sin(u) du \\ &= u^2 \sin(u) + 2u \cos(u) - 2 \sin(u). \end{aligned}$$

We used part a to determine the integral on the right.

1. i. $F(x) = x, g(x) = \sec^2(x)$ (so $G(x) = \tan(x)$);

$$\begin{aligned} \int x \sec^2(x) dx &= x \tan(x) - \int \tan(x) dx \\ &= x \tan(x) + \ln |\cos(x)|. \end{aligned}$$

1. j. First write

$$\int e^{2x}(x + e^x) dx = \int x e^{2x} + e^{3x} dx = \int x e^{2x} dx + \frac{1}{3} e^{3x},$$

and use integration by parts on the remaining integral: $F(x) = x, g(x) = e^{2x}$;

$$\int x e^{2x} dx = \frac{x e^{2x}}{2} - \frac{e^{2x}}{4}.$$

Thus the original integral is

$$\frac{e^{3x}}{3} + \frac{x e^{2x}}{2} - \frac{e^{2x}}{4} = \frac{e^{2x}}{12} (6x - 3 + 4e^x).$$

2. $F(x) = \ln x$, $g(x) = \ln x$, so $G(x) = x \ln x - x$ from page 710 of the text, and also from exercise 15b in section 11.1. Therefore,

$$\begin{aligned}\int (\ln x)^2 dx &= (\ln x)(x \ln x - x) - \int \frac{x \ln x - x}{x} dx \\ &= x(\ln x)^2 - x \ln x - \int (\ln x - 1) dx \\ &= x(\ln x)^2 - 2x \ln x + 2x.\end{aligned}$$

3. a. Let $F(x) = x^2$, $g(x) = e^x$;

$$\begin{aligned}\int x^2 e^x dx &= x^2 e^x - 2 \int x e^x dx \\ &= x^2 e^x - 2x e^x + 2e^x = e^x(x^2 - 2x + 2).\end{aligned}$$

The integral on the right is essentially the same as in exercise 1b.

3. b. Let $F(x) = x^3$, $g(x) = e^x$;

$$\int x^3 e^x dx = x^3 e^x - 3 \int x^2 e^x dx$$

The integral on the right is from part a; we therefore find

$$\begin{aligned}\int x^3 e^x dx &= x^3 e^x - 3e^x(x^2 - 2x + 2) \\ &= e^x(x^3 - 3x^2 + 6x - 6).\end{aligned}$$

3. c. Let $F(x) = x^4$, $g(x) = e^x$;

$$\int x^4 e^x dx = x^4 e^x - 4 \int x^3 e^x dx$$

The integral on the right is from part b; we therefore find

$$\begin{aligned}\int x^4 e^x dx &= x^4 e^x - 4e^x(x^3 - 3x^2 + 6x - 6) \\ &= e^x(x^4 - 4x^3 + 12x^2 - 24x + 24).\end{aligned}$$

3. d. Here is the general pattern

$$\int x^n e^x dx = e^x P_n(x),$$

where $P_n(x)$ is a polynomial of degree n . Now $(e^x P_n)' = e^x x^n$; expanding this we get $e^x P_n + e^x P_n' = e^x x^n$, or just $P_n + P_n' = x^n$. This is just a differential equation for $y = P_n(x)$ (namely $y + y' = x^n$), and we can solve it the same way we found power series solutions to differential equations.

Let $y = P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} + a_n x^n$. Then

$$\begin{aligned}y + y' &= (a_0 + a_1) + (a_1 + 2a_2)x + (a_2 + 3a_3)x^2 + \cdots \\ &\quad + (a_{n-1} + na_n)x^{n-1} + a_n x^n.\end{aligned}$$

When we set $y + y' = x^n$ we get the equations

$$\begin{aligned}a_n &= 1 \\ a_{n-1} + na_n &= 0 \\ a_{n-2} + (n-1)a_{n-1} &= 0 \\ &\vdots \\ a_2 + 3a_3 &= 0 \\ a_1 + 2a_2 &= 0 \\ a_0 + a_1 &= 0\end{aligned}$$

Since $a_n = 1$, we work from the “top down” to get the other coefficients recursively:

$$\begin{aligned}a_{n-1} &= -na_n = -n \\ a_{n-2} &= -(n-1)a_{n-1} = n(n-1) \\ &\vdots \\ a_2 &= -3a_3 = (-1)^{n-2}n(n-1)\cdots 3 \\ a_1 &= -2a_2 = (-1)^{n-1}n(n-1)\cdots 3\cdot 2 \\ a_0 &= -a_1 = (-1)^n n(n-1)\cdots 3\cdot 2\cdot 1\end{aligned}$$

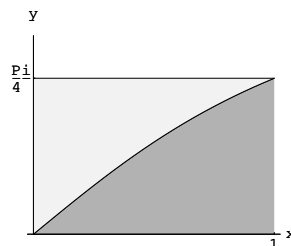
We can write these in the form $a_j = (-1)^{n-j} n! / j!$. Thus

$$\int x^n e^x dx = e^x \sum_{j=0}^n (-1)^{n-j} \frac{n!}{j!} x^j.$$

3. e. Break up the integral:

$$\begin{aligned}\int e^x (5x^2 - 3x + 7) dx &= 5 \int x^2 e^x dx - 3 \int x e^x dx + 7 \int e^x dx \\ &= 5e^x(x^2 - 2x + 2) - 3e^x(x - 1) + 7e^x \\ &= e^x(5x^2 - 13x + 20)\end{aligned}$$

4. a.



The corresponding relation is $x = \tan(y)$ over the interval $0 \leq y \leq \pi/4$.

4. b. By exercise 1f, above,

$$\begin{aligned}\int_0^1 \arctan(x) dx &= \left(x \arctan x - \frac{\ln(1+x^2)}{2} \right) \Big|_0^1 \\ &= \frac{\pi}{4} - \frac{\ln(2)}{2}.\end{aligned}$$

This corresponds to the darker shaded area in the figure in part a.

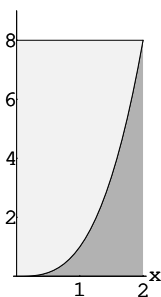
4. c. From exercise 1g in section 11.2 we find

$$\begin{aligned}\int_0^{\pi/4} \tan(y) dy &= -\ln |\cos(y)| \Big|_0^{\pi/4} \\ &= \frac{-\ln(1/2) + \ln(1)}{2} = \frac{\ln(2)}{2}.\end{aligned}$$

This corresponds to the more lightly shaded area in the figure in part a.

4. d. The sum of the two integrals is $\pi/4$. This should equal the sum of the two shaded areas. In fact, the two shaded areas together make up the rectangle of dimensions $1 \times \pi/4$. So the sum of the integrals *should be* $\pi/4$, and it is.

5. y



$$\int_0^2 x^3 dx \quad \text{dark}$$

$$\int_0^8 y^{1/3} dy \quad \text{light}$$

$$\int_0^2 x^3 dx + \int_0^8 y^{1/3} dy = \frac{x^4}{4} \Big|_0^2 + \frac{3y^{4/3}}{4} \Big|_0^8 = 4 + 12 = 16.$$

This is just what we would expect, because the area of the whole shaded rectangle is 16.

6. Suppose $y = f(x)$ and $x = g(y)$ are inverses, and suppose also that $f(0) = g(0) = 0$ and $b = f(a)$ so $a = g(b)$. Then

$$\int_0^a f(x) dx + \int_0^b g(y) dy = ab.$$

$$\begin{aligned}\mathbf{7. a.} \quad & \frac{1}{e-1} \int_1^e \ln(x) dx \\ &= \frac{x \ln(x) - x}{e-1} \Big|_1^e = \frac{e \ln(e) - e + 1}{e-1} = \frac{1}{e-1}.\end{aligned}$$

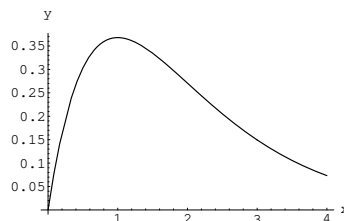
$$\begin{aligned}\mathbf{7. b.} \quad & \frac{1}{b-1} \int_1^b \ln(x) dx \\ &= \frac{x \ln(x) - x}{b-1} \Big|_1^b = \frac{b \ln(b) - b + 1}{b-1}.\end{aligned}$$

If we divide the numerator and denominator of this last expression by b we get

$$\frac{\ln(b) - 1 + 1/b}{1 - 1/b} \approx \ln(b) - 1 \quad \text{when } b \text{ is large.}$$

The last step can be justified on the grounds that $1/b \approx 0$ when b is large.

8. a.



$$\mathbf{8. b.} \quad \int_0^4 x e^{-x} dx = -x e^{-x} - e^{-x} \Big|_0^4 = 1 - 5e^{-4}.$$

$$\begin{aligned}\mathbf{8. c.} \quad A(b) &= \int_0^b x e^{-x} dx = -x e^{-x} - e^{-x} \Big|_0^b \\ &= 1 - (b+1)e^{-b}. \\ A(100) &= 1 - 101/e^{100} = 1 - (3.75728 \times 10^{-42}), \text{ so} \\ A(100) &= 1 \text{ to 41 decimal places.}\end{aligned}$$

9. All solutions are of the form $\int 5 - 2 \ln(t) dt$

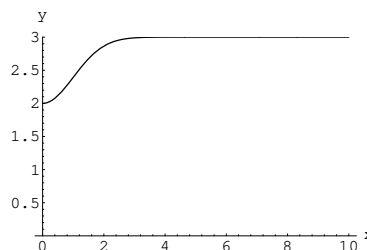
$$= 5t - 2(t \ln(t) - t) + C = 7t - 2t \ln(t) + C;$$

C can take any value.

10. a. Since $\varphi(0) = 2$ and $\varphi'(t) = t e^{-t^2/2}$,

$$\begin{aligned}\varphi(t) &= 2 + \int_0^t z e^{-z^2/2} dz = 2 - e^{-z^2/2} \Big|_0^t \\ &= 2 - e^{-t^2/2} + 1 = 3 - e^{-t^2/2}.\end{aligned}$$

10. b.



Since $\varphi'(t) = t e^{-t^2/2} > 0$ when $t > 0$, $\varphi(t)$ is an increasing function.

10. c. The graph indicates that $\varphi(t)$ does *not* increase without bound as $t \rightarrow \infty$. The formula makes it clear that $\varphi(t) < 3$ for all $t \geq 0$; furthermore, $\varphi(t) \rightarrow 3$ as $t \rightarrow \infty$.

11. a. The equation $\frac{d}{dx}(u \cdot v) = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$ implies

$$uv = \int \frac{d}{dx}(u \cdot v) dx = \int v \cdot \frac{du}{dx} dx + \int u \cdot \frac{dv}{dx} dx.$$

But since $du = \frac{du}{dx} dx$, and similarly for dv , we finally have

$$uv = \int v du + \int u dv,$$

which is algebraically equivalent to the desired formula.

11. b. *Ad libitum*; the answer depends on choices made by the student.

Sine and Cosine Integrals

12. a. *Ad libitum*. For any α , the graphs of $2\sin \alpha x \cos \alpha x$ and $\sin 2\alpha x$ should have amplitude 1, period π/α , and pass through the origin.

For any α , the graphs of $\cos^2 \alpha x - \sin^2 \alpha x$ and $\cos 2\alpha x$ should have amplitude 1, period π/α , and hit their maximum of 1 at $x = 0$.

The graph of $\sin^2 \alpha x + \cos^2 \alpha x$ should be identically 1 for any α .

12. b. When $f(x) = 2\sin \alpha x \cos \alpha x$, then

$$f'(x) = 2\alpha(\cos^2 \alpha x - \sin^2 \alpha x)$$

$$f''(x) = -8\alpha^2 \sin \alpha x \cos \alpha x,$$

so $f'' = -4\alpha^2 f$. Also $f(0) = 0$, $f'(0) = 2\alpha$.

When $g(x) = \sin 2\alpha x$, then

$$g'(x) = 2\alpha \cos 2\alpha x \quad g''(x) = -4\alpha^2 \sin 2\alpha x,$$

so $g'' = -4\alpha^2 g$. Also $g(0) = 0$, $g'(0) = 2\alpha$. Thus f and g solve the same initial value problem. Since initial value problems have unique solutions, $f(x) = g(x)$ for all x .

12. c. It is a bit easier to work with $g(x) = \cos 2\alpha x$. We have

$$g'(x) = -2\alpha \sin 2\alpha x \quad g''(x) = -4\alpha^2 \cos 2\alpha x$$

and $g(0) = 1$, $g'(0) = 0$. Therefore g is a solution to the initial value problem

$$y'' = -4\alpha^2 y \quad \text{with } y(0) = 1 \text{ and } y'(0) = 0.$$

When $f(x) = \cos^2 \alpha x - \sin^2 \alpha x$, then

$$f'(x) = -2\alpha \sin \alpha x \cos \alpha x$$

$$f''(x) = -4\alpha^2(\cos^2 \alpha x - \sin^2 \alpha x)$$

and $f(0) = 1$, $f'(0) = 0$. Therefore f solves the same initial value problem as g and must equal g .

Note: This is not the only possibility. The differential equation can't change but the initial conditions can. For example, the initial conditions could have been $y(\pi/4\alpha) = 0$, $y'(\pi/4\alpha) = 0$.

13. See the solution to exercise 14. This is just the special case $\alpha = 1$.

14. a. Let $F(x) = \sin \alpha x$, $g(x) = \sin \alpha x$; then $f(x) = \alpha \cos \alpha x$, $G(x) = -\cos \alpha x/\alpha$ and

$$\int \sin^2 \alpha x dx = -\frac{\sin \alpha x \cos \alpha x}{\alpha} + \int \cos^2 \alpha x dx.$$

$$\mathbf{14. b.} \quad \int \cos^2 \alpha x dx =$$

$$\int 1 - \sin^2 \alpha x dx = x - \int \sin^2 \alpha x dx.$$

14. c. Combining parts a and b, we get

$$\begin{aligned} \int \sin^2 \alpha x dx = \\ -\frac{\sin \alpha x \cos \alpha x}{\alpha} + x - \int \sin^2 \alpha x dx. \end{aligned}$$

If we bring the rightmost integral over to the other side of this equation and use the identity $\sin \alpha x \cos \alpha x = \sin 2\alpha x/2$, we get

$$2 \int \sin^2 \alpha x dx = x - \frac{\sin 2\alpha x}{2\alpha},$$

or just

$$\int \sin^2 \alpha x dx = \frac{x}{2} - \frac{\sin 2\alpha x}{4\alpha}.$$

15. a. This can be done a number of ways. Here's one: $1 - \cos 2\alpha x$ equals

$$\begin{aligned} &= \sin^2 \alpha x + \cos^2 \alpha x - (\cos^2 \alpha x - \sin^2 \alpha x) \\ &= 2\sin^2 \alpha x \end{aligned}$$

$$\mathbf{15. b.} \quad \int \sin^2 \alpha x dx = \int \frac{1 - \cos 2\alpha x}{2} dx$$

$$= \int \frac{1}{2} dx - \frac{1}{2} \int \cos 2\alpha x dx = \frac{x}{2} - \frac{1}{4\alpha} \sin 2\alpha x.$$

The last integral was done by substitution, with $u = 2\alpha x$, $du = 2\alpha dx$, $dx = du/2\alpha$.

$$\begin{aligned}
 16. \quad \int \cos^2 \alpha x dx &= \int (1 - \sin^2 \alpha x) dx \\
 &= x - \left(\frac{x}{2} - \frac{1}{4\alpha} \sin 2\alpha x \right) = \frac{x}{2} + \frac{1}{4\alpha} \sin 2\alpha x.
 \end{aligned}$$

$$\begin{aligned}
 17. \text{ a. } \quad \int \sin \alpha x \cos \alpha x dx &= \frac{1}{2} \int \sin 2\alpha x dx \\
 &= \frac{1}{2} \left(\frac{-\cos 2\alpha x}{2\alpha} \right) = -\frac{1}{4\alpha} \cos 2\alpha x
 \end{aligned}$$

17. b. Let $u = \sin \alpha x$; then $du = \alpha \cos \alpha x dx$ and

$$\int \sin \alpha x \cos \alpha x dx = \frac{1}{\alpha} \int u du = \frac{u^2}{2\alpha} = \frac{\sin^2 \alpha x}{2\alpha}$$

17. c. In exercise 15a we find $\sin^2 \alpha x = (1 - \cos 2\alpha x)/2$. Therefore,

$$\begin{aligned}
 \frac{1}{2\alpha} \sin^2 \alpha x + C &= \frac{1}{2\alpha} \left(\frac{1 - \cos 2\alpha x}{2} \right) + C \\
 &= \frac{1}{4\alpha} - \frac{1}{4\alpha} \cos 2\alpha x + C = -\frac{1}{4\alpha} \cos 2\alpha x + C',
 \end{aligned}$$

where C' is the new constant $C + \frac{1}{4\alpha}$. Therefore the solutions to parts a and b are compatible.

18. a. Let $F(x) = \sin \beta x$, $g(x) = \sin \alpha x$. Then $f(x) = \beta \cos \beta x$, $G(x) = -\cos \alpha x / \alpha$, and

$$\begin{aligned}
 \int \sin \alpha x \sin \beta x dx \\
 &= -\frac{\cos \alpha x \sin \beta x}{\alpha} + \frac{\beta}{\alpha} \int \cos \alpha x \cos \beta x dx.
 \end{aligned}$$

18. b. Let $F(x) = \cos \beta x$, $g(x) = \cos \alpha x$. Then $f(x) = -\beta \sin \beta x$, $G(x) = \sin \alpha x / \alpha$ and

$$\begin{aligned}
 \int \cos \alpha x \cos \beta x dx \\
 &= \frac{\sin \alpha x \cos \beta x}{\alpha} + \frac{\beta}{\alpha} \int \sin \alpha x \sin \beta x dx.
 \end{aligned}$$

Notice that the integral on the right is just J , as defined in the next part of this exercise.

$$\begin{aligned}
 18. \text{ c. } \quad J &= -\frac{\cos \alpha x \sin \beta x}{\alpha} + \frac{\beta}{\alpha} \int \cos \alpha x \cos \beta x dx \\
 &= -\frac{\cos \alpha x \sin \beta x}{\alpha} + \frac{\beta}{\alpha} \left(\frac{\sin \alpha x \cos \beta x}{\alpha} + \frac{\beta}{\alpha} J \right) \\
 &= -\frac{1}{\alpha} \cos \alpha x \sin \beta x + \frac{\beta}{\alpha^2} \sin \alpha x \cos \beta x + \frac{\beta^2}{\alpha^2} J.
 \end{aligned}$$

$$\begin{aligned}
 18. \text{ d. } \quad \left(1 - \frac{\beta^2}{\alpha^2} \right) J &= \frac{\beta}{\alpha^2} \sin \alpha x \cos \beta x - \frac{1}{\alpha} \cos \alpha x \sin \beta x. \\
 \text{Since } 1 - \frac{\beta^2}{\alpha^2} &= \frac{\alpha^2 - \beta^2}{\alpha^2}, \text{ we can solve for } J \text{ and get}
 \end{aligned}$$

$$\begin{aligned}
 J &= \frac{\alpha^2}{\alpha^2 - \beta^2} \left(\frac{\beta}{\alpha^2} \sin \alpha x \cos \beta x - \frac{1}{\alpha} \cos \alpha x \sin \beta x \right) \\
 &= \frac{1}{\beta^2 - \alpha^2} (\alpha \cos \alpha x \sin \beta x - \beta \sin \alpha x \cos \beta x).
 \end{aligned}$$

Note the reversal $1/(\beta^2 - \alpha^2) = -1/(\alpha^2 - \beta^2)$ in the last step.

19. a. Let $F(x) = \cos \beta x$, $g(x) = \cos \alpha x$. Then

$$\begin{aligned}
 \int \cos \alpha x \cos \beta x dx \\
 &= \frac{\sin \alpha x \cos \beta x}{\alpha} + \frac{\beta}{\alpha} \int \sin \alpha x \sin \beta x dx
 \end{aligned}$$

If we take the expression for the integral on the right from the solution to exercise 18a, above, and define $J = \int \cos \alpha x \cos \beta x dx$, then we have

$$\begin{aligned}
 J &= \frac{\sin \alpha x \cos \beta x}{\alpha} + \frac{\beta}{\alpha} \left(-\frac{\cos \alpha x \sin \beta x}{\alpha} + \frac{\beta}{\alpha} J \right) \\
 &= \frac{1}{\alpha} \sin \alpha x \cos \beta x - \frac{\beta}{\alpha^2} \cos \alpha x \sin \beta x + \frac{\beta^2}{\alpha^2} J
 \end{aligned}$$

Therefore

$$\left(1 - \frac{\beta^2}{\alpha^2} \right) J = \frac{1}{\alpha} \sin \alpha x \cos \beta x - \frac{\beta}{\alpha^2} \cos \alpha x \sin \beta x,$$

and solving for J we get

$$\begin{aligned}
 J &= \frac{\alpha^2}{\alpha^2 - \beta^2} \left(\frac{1}{\alpha} \sin \alpha x \cos \beta x - \frac{\beta}{\alpha^2} \cos \alpha x \sin \beta x \right) \\
 &= \frac{1}{\beta^2 - \alpha^2} (\beta \cos \alpha x \sin \beta x - \alpha \sin \alpha x \cos \beta x).
 \end{aligned}$$

19. b. To analyze the second integral, let $F(x) = \cos \beta x$, $g(x) = \sin \alpha x$. Then

$$\begin{aligned}
 \int \sin \alpha x \cos \beta x dx \\
 &= -\frac{\cos \alpha x \cos \beta x}{\alpha} - \frac{\beta}{\alpha} \int \cos \alpha x \sin \beta x dx
 \end{aligned}$$

To determine the integral on the right, let $F(x) = \sin \beta x$, $g(x) = \cos \alpha x$. Then

$$\begin{aligned}
 \int \cos \alpha x \sin \beta x dx \\
 &= \frac{\sin \alpha x \sin \beta x}{\alpha} - \frac{\beta}{\alpha} \int \sin \alpha x \cos \beta x dx
 \end{aligned}$$

Let $J = \int \sin \alpha x \cos \beta x dx$; combining the last two equations, we have

$$\begin{aligned} J &= -\frac{\cos \alpha x \cos \beta x}{\alpha} - \frac{\beta}{\alpha} \left(\frac{\sin \alpha x \sin \beta x}{\alpha} - \frac{\beta}{\alpha} J \right) \\ &= -\frac{1}{\alpha} \cos \alpha x \cos \beta x - \frac{\beta}{\alpha^2} \sin \alpha x \sin \beta x + \frac{\beta^2}{\alpha^2} J \end{aligned}$$

Therefore

$$\left(1 - \frac{\beta^2}{\alpha^2}\right) J = -\frac{1}{\alpha} \cos \alpha x \cos \beta x - \frac{\beta}{\alpha^2} \sin \alpha x \sin \beta x,$$

and solving for J we get

$$\begin{aligned} J &= \frac{\alpha^2}{\alpha^2 - \beta^2} \left(-\frac{1}{\alpha} \cos \alpha x \cos \beta x - \frac{\beta}{\alpha^2} \sin \alpha x \sin \beta x \right) \\ &= \frac{1}{\beta^2 - \alpha^2} (\beta \sin \alpha x \sin \beta x + \alpha \cos \alpha x \cos \beta x). \end{aligned}$$

20. This is purely a matter of substituting β for α , α for β , and doing the algebra.

21. a. Use the solution to exercise 13:

$$\int_0^{2\pi} \sin^2 x dx = \frac{x}{2} - \frac{\sin 2x}{4} \Big|_0^{2\pi} = \pi.$$

21. b. Use the solution to exercise 16:

$$\int_0^{n\pi} \cos^2 x dx = \frac{x}{2} + \frac{\sin 2x}{4} \Big|_0^{n\pi} = \frac{n\pi}{2}.$$

21. c. Use the solution to exercise 18 (with $\alpha = 1$):

$$\begin{aligned} \int_0^{2\pi} \sin x \sin \beta x dx &= \frac{\cos x \sin \beta x - \beta \sin x \cos \beta x}{\beta^2 - 1} \Big|_0^{2\pi} = \frac{\sin 2\pi \beta}{\beta^2 - 1}. \end{aligned}$$

21. d. $\int_0^\pi \sin x \sin \beta x dx =$

$$\frac{\cos x \sin \beta x - \beta \sin x \cos \beta x}{\beta^2 - 1} \Big|_0^\pi = \frac{\sin \pi \beta}{1 - \beta^2}.$$

11.4 Separation of Variables and Partial Fractions

Separation of Variables

1. Replace $dy/dt = y + 5$ by $dt = dy/(y + 5)$; then

$$t = \int \frac{1}{y+5} dy = \ln|y+5| + k.$$

or $t + b = \ln|y + 5|$, where $b = -k$. Now solve for y in terms of t :

$$y + 5 = \pm e^{t+b} = \pm e^b \cdot e^t = Ce^t,$$

where $C = \pm e^b$. Therefore $y = Ce^t - 5$. (Note that e^b is always positive and thus is not completely arbitrary. However, $C = \pm e^b$ is arbitrary. We can even have $C = 0$, which is clear from the original differential equation.)

2. a. $dt = y dy$, so $t = \int y dy = \frac{y^2}{2} + k$. Now solve for y :

$$\frac{y^2}{2} = t - k, \quad y = \pm \sqrt{2t - 2k} = \pm \sqrt{2t + C},$$

where $C = -2k$.

2. b. $3 dx = (z - 2) dz$, so

$$3x = \int z - 2 dz = \frac{1}{2}(z - 2)^2 + k.$$

Now solve for z :

$$(z - 2)^2 = 6x - 2k, \quad z - 2 = \pm \sqrt{6x - 2k} = \pm \sqrt{6x + C},$$

where $C = -2k$. Therefore $z = 2 \pm \sqrt{6x + C}$.

2. c. $y dy = x dx$, so

$$\frac{y^2}{2} = \int y dy = \int x dx = \frac{x^2}{2} + k,$$

and thus

$$y = \pm \sqrt{x^2 + 2k} = \pm \sqrt{x^2 + C}$$

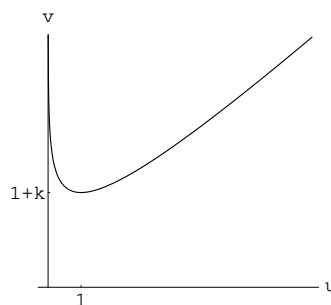
2. d. $dy/y = dx/x$, so

$$\ln|y| = \int \frac{dy}{y} = \int \frac{dx}{x} = \ln|x| + k,$$

or $|y| = e^k|x|$. This can be rewritten $y = Cx$ where $C = \pm e^k$, and C is arbitrary.

2. e. $dv = \frac{u-1}{u} du = 1 - \frac{1}{u} du$. Therefore

$$v = \int 1 - \frac{1}{u} du = u - \ln(u) + k \quad \text{for } u > 0.$$



We need to solve for u in terms of v . From the graph of $v = u - \ln(u) + k$ (shown above with $k = 2$), it appears that there are two functions $u = f_1(v)$, $u = f_2(v)$ that serve as inverses. Each is defined for $v \geq 1 + k$, and $f_1(1 + k) = f_2(1 + k) = 1$. To distinguish between f_1 and f_2 we can say $f_1(v) \rightarrow 0$ as $v \rightarrow \infty$ while $f_2(v) \approx v$ when v is large. However, there are no quick formulas for f_1 and f_2 .

2. f. $-dt = dv/\sqrt{v}$. Therefore

$$-t = 2\sqrt{v} + k, \quad v = \frac{(C-t)^2}{4},$$

where $C = -k$.

3. a. $-k dt = dQ/(Q - C)$, so

$$-kt = \int \frac{dQ}{Q-C} = \ln|Q-C| + b,$$

where b is an arbitrary constant. (The constants k and C are *not* arbitrary; on the contrary, they are determined by the physical situation.) Now solve for Q :

$$\ln|Q-C| = -kt - b, \quad Q-C = \pm e^{-kt-b},$$

so $Q = C + \alpha e^{-kt}$. Here $\alpha = \pm e^{-b}$ is an arbitrary constant; it can be positive, negative, or zero.

3. b. With the given information we have $Q(t) = 20 + \alpha e^{-.1t}$ and $90 = Q(0) = 20 + \alpha$. Therefore $\alpha = 70$ and

$$Q(t) = 20 + 70e^{-.1t}, \quad Q(20) = 20 + 70e^{-2} = 29.4735.$$

After 20 minutes the temperature will be about $29\frac{1}{2}^\circ\text{C}$.

3. c. We must solve $Q(t) = 30$ for t :

$$30 = 20 + 70e^{-.1t}, \quad 10 = 70e^{-.1t}, \quad -.1t = \ln(10/70),$$

or $t = -10\ln(1/7) = 19.459$ minutes.

4. a. $C = 90^\circ\text{F}$. and $k = .2^\circ\text{F}$. per minute/ $^\circ\text{F}$. The differential equation is $Q' = -.2(Q - 90)$.

4. b. $Q(t) = 90 + \alpha e^{-.2t}$, by analogy with the solution to the previous exercise. Since $36 = Q(0) = 90 + \alpha$, $\alpha = -54$ and the solution is

$$Q(t) = 90 - 54e^{-.2t}.$$

4. c. $Q(5) = 90 - 54e^{-1} \approx 70.13^\circ\text{F}$.
 $Q(10) = 90 - 54e^{-2} \approx 82.69^\circ\text{F}$.

4. d. We must solve for t in $55 = Q(t) = 90 - 54e^{-.2t}$. Thus $35/54 = e^{-.2t}$, so $-.2t = \ln(35/54)$ and $t = 2.17$ minutes.

5. a. This is similar to exercise 2f: $-k dt = dV/\sqrt{V}$, so

$$-kt = 2\sqrt{V} + b, \quad \sqrt{V} = \frac{-kt - b}{2} = \frac{k}{2}(C - t),$$

where $C = -2b/k$ is arbitrary. Therefore

$$V = \frac{k^2}{4}(C - t)^2 \quad \text{for any } C.$$

5. b. First of all, when $0 \leq t \leq C$ we have $V'(t) = -k^2(C - t)/2$ and $\sqrt{V} = k(C - t)/2$; therefore $V' = -k\sqrt{V}$. Next, when $C < t$ we have $V'(t) = 0$ and $V(t) = 0$ so $V' = -k\sqrt{V}$ as well. Thus the given function is a solution to the differential equation.

This solution is more relevant to the leaking tank problem because it says that the tank *stays* empty once it becomes empty; that's the meaning of $V(t) = 0$ for all $t \geq C$.

6. a. $-dt = dv/(g + bv)$, so

$$-t + k = \frac{\ln|g + bv|}{b}, \quad \ln|g + bv| = -bt + kb.$$

Therefore $g + bv = Ce^{-bt}$, where $C = \pm e^{kb}$ is arbitrary (because b was). The final form is therefore

$$v(t) = \frac{1}{b}(Ce^{-bt} - g).$$

6. b. If $0 = v(0) = (C - g)/b$, then $C = g$ and so

$$v(t) = \frac{g}{b}(e^{-bt} - 1).$$

6. c. Since $2 = e^{\ln(2)}$, we have $e = 2^{1/\ln(2)} \approx 2^{1/.69}$. Therefore, we can replace e in the expression just obtained for $v(t)$ and get

$$v(t) = \frac{g}{b}(2^{-bt/.69} - 1).$$

This is how $v(t)$ is given in exercise 21 in section 4.2.

$$\begin{aligned} \mathbf{6. d.} \quad x(t) &= \int_0^t \frac{g}{b}(e^{-bt} - 1) dt = -\frac{g}{b^2}e^{-bt} - \frac{g}{b}t \Big|_0^t = \\ &= -\frac{g}{b^2}t - \frac{g}{b^2}(e^{-bt} - 1), \end{aligned}$$

which is in a form consistent with the solution given in exercise 22 in section 4.2.

7. a. $k dt = dQ/Q^2$, so

$$kt + b = -\frac{1}{Q}, \quad Q = \frac{1}{C - kt},$$

where $C = -b$ is arbitrary.

7. b. If $A = Q(0) = 1/C$, then $C = 1/A$ and

$$Q(t) = \frac{1}{1/A - kt} = \frac{A}{1 - kAt}.$$

7. c. $Q(t)$ becomes infinite when its denominator becomes zero. Therefore $\infty = Q(\tau) = A/(1 - kA\tau)$ when $\tau = 1/kA$. Thus, if either the growth constant k or the initial population A is larger, the explosion to infinity happens sooner.

7. d. The given information is that $\Delta k/k = \Delta A/A = \pm 5\%$. Our aim is to determine $\frac{\Delta \tau}{\tau}$. Consider $\tau = 1/kA$ first as a function of k . An error Δk in k propagates to τ by

$$\Delta \tau \approx \frac{d\tau}{dk} \Delta k = -\frac{1}{k^2 A} \Delta k.$$

The relative error is

$$\frac{\Delta \tau}{\tau} \approx \frac{-1/k^2 A}{1/kA} \Delta k = -\frac{\Delta k}{k} = \mp 5\%$$

Now carry out the same analysis on $\tau = 1/kA$ as a function of A . In fact, since k and A appear the same way in the expression for τ , the result will be the same:

$$\frac{\Delta \tau}{\tau} \approx -\frac{\Delta A}{A} = \mp 5\%.$$

In other words, a 5% error in *either* k or A will produce a 5% error in τ ; the errors (or uncertainties) are the same.

8. We assume $p \neq 1$, because that case is just ordinary exponential growth. Separation of variables gives $k dt = Q^{-p} dQ$, so

$$kt + b = \frac{Q^{1-p}}{1-p}, \quad Q = ((1-p)kt + C)^{1/1-p},$$

where $C = (1-p)b$ is arbitrary. However, the initial condition $Q(0) = A$ determines C , as follows: $A = Q(0) = C^{1/1-p}$, so $C = A^{1-p}$ and

$$Q(t) = ((1-p)kt + A^{1-p})^{1/1-p}.$$

We will have $Q(\tau) = \infty$ if and only if the exponent $1/1-p$ is negative and the expression $(1-p)k\tau + A^{1-p}$ is zero. The former condition translates into $1-p < 0$ or $1 < p$. The latter condition is

$$\tau = \frac{-A^{1-p}}{(1-p)k} = \frac{A^{1-p}}{(p-1)k} > 0.$$

Partial Fractions

9. a. $1 = \alpha(x+2) + \beta(x-1) = (\alpha + \beta)x + (2\alpha - \beta).$

$$\begin{aligned} 0 &= \alpha + \beta & \alpha &= 1/3 \\ 1 &= 2\alpha - \beta & \beta &= -1/3 \end{aligned}$$

9. b. $x = (\alpha + \beta)x + (2\alpha - \beta).$

$$\begin{aligned} 1 &= \alpha + \beta & \alpha &= 1/3 \\ 0 &= 2\alpha - \beta & \beta &= 2/3 \end{aligned}$$

9. c. $1 = \alpha(x^2 - 1) + \beta x(x+1) + \gamma x(x-1) = (\alpha + \beta + \gamma)x^2 + (\beta - \gamma)x - \alpha.$

$$\begin{aligned} 0 &= \alpha + \beta + \gamma & \alpha &= -1 \\ 0 &= \beta - \gamma & \beta &= 1/2 \\ 1 &= -\alpha & \gamma &= 1/2 \end{aligned}$$

9. d. $x = \alpha(x+1) + \beta(2x+1) = (\alpha + 2\beta)x + \alpha + \beta.$

$$\begin{aligned} 1 &= \alpha + 2\beta & \alpha &= -1 \\ 0 &= \alpha + \beta & \beta &= 1 \end{aligned}$$

9. e. $1 = \alpha(x^2 + 1) + (\beta x + \gamma)x = (\alpha + \beta)x^2 + \gamma x + \alpha.$

$$\begin{aligned} 0 &= \alpha + \beta & \alpha &= 1 \\ 0 &= \gamma & \beta &= -1 \\ 1 &= \alpha & \gamma &= 0 \end{aligned}$$

10. a. Use the solution to exercise 9a:

$$\int \frac{3dx}{(x-1)(x+2)} = \int \frac{dx}{x-1} - \int \frac{dx}{x+2} = \ln \left| \frac{x-1}{x+2} \right|.$$

10. b. Multiply the solution to exercise 9b by 5, multiply the solution to exercise 9a by 3, and add:

$$\begin{aligned} \int \frac{5x+3}{(x-1)(x+2)} dx &= \frac{8}{3} \int \frac{dx}{x-1} + \frac{7}{3} \int \frac{dx}{x+2} \\ &= \frac{8}{3} \ln|x-1| + \frac{7}{3} \ln|x+2|. \end{aligned}$$

10. c. Use the solution to exercise 9c:

$$\begin{aligned} \int \frac{dt}{t(t^2-1)} &= -\int \frac{dt}{t} + \frac{1}{2} \int \frac{dt}{t+1} + \frac{1}{2} \int \frac{dt}{t-1} \\ &= -\ln|t| + \frac{1}{2} \ln|t+1| + \frac{1}{2} \ln|t-1| \\ &= \ln \left| \sqrt{1 - \frac{1}{t^2}} \right|. \end{aligned}$$

$$\begin{aligned} 10. \text{ d. } \int \frac{x dx}{1-x^2} &= -\frac{1}{2} \int \frac{dx}{1+x} + \frac{1}{2} \int \frac{dx}{1-x} \\ &= -\frac{1}{2} \ln|1+x| - \frac{1}{2} \ln|1-x|. \end{aligned}$$

$$\begin{aligned} 10. \text{ e. } \int \frac{1-u}{u^2-4} du &= -\frac{3}{4} \int \frac{du}{u+2} - \frac{1}{4} \int \frac{du}{u-2} = \\ &= -\frac{3}{4} \ln|u+2| - \frac{1}{4} \ln|u-2|. \end{aligned}$$

10. f. Vary exercise 9e with $x^2 + 2x + 1 = (\alpha + \beta)x^2 + \gamma x + \alpha$ to get $\alpha = 1$, $\beta = 0$, $\gamma = 2$, so

$$\begin{aligned} \int \frac{x^2 + 2x + 1}{x(x^2 + 1)} dx &= \int \frac{dx}{x} + 2 \int \frac{dx}{x^2 + 1} \\ &= \ln|x| + 2 \arctan x. \end{aligned}$$

$$11. \text{ a. } \int_2^3 \frac{3 dx}{(x-1)(x+2)} = \ln \left| \frac{x-1}{x+2} \right| \Big|_2^3 = \ln \frac{8}{5}.$$

$$\begin{aligned} 11. \text{ b. } \int_2^4 \frac{dt}{t(t^2-1)} &= \ln \sqrt{\left| \frac{t^2-1}{t^2} \right|} \Big|_2^4 \\ &= \ln \sqrt{\frac{15}{16}} - \ln \sqrt{\frac{3}{4}} = \ln \sqrt{\frac{15}{16} \cdot \frac{4}{3}} = \ln \sqrt{\frac{5}{4}} = \ln \frac{\sqrt{5}}{2}. \end{aligned}$$

$$\begin{aligned} 11. \text{ c. } \int_0^{\pi/4} \frac{x dx}{1-x^2} &= -\frac{1}{2} \ln|1-x^2| \Big|_0^{\pi/4} \\ &= -\frac{1}{2} \ln(1 - \pi^2/16) = \ln \sqrt{\frac{16}{16 - \pi^2}}. \end{aligned}$$

$$\begin{aligned} 11. \text{ d. } \int_1^{\sqrt{3}} \frac{x^2 + 2x + 1}{x(x^2 + 1)} dx &= \ln|x| + 2 \arctan x \Big|_1^{\sqrt{3}} \\ &= \ln \sqrt{3} + 2 \frac{\pi}{3} - 2 \frac{\pi}{4} = \ln \sqrt{3} + \frac{\pi}{6}. \end{aligned}$$

$$12. \sec x = \frac{1}{\cos x} = \frac{\cos x}{\cos^2 x} = \frac{\cos x}{1 - \sin^2 x}, \text{ so}$$

$$\int \sec x dx = \int \frac{\cos x dx}{1 - \sin^2 x} = \int \frac{du}{1 - u^2},$$

where $u = \sin x$, $du = \cos x dx$. The last integral is the same as the one the text analyzed on page 727, so we needn't repeat that work here. Note however that, because

this integral is the negative of the earlier one, the numerator and denominator in the logarithm are reversed.

$$\int \frac{du}{1-u^2} = \frac{1}{2} \ln \frac{1+u}{1-u} = \frac{1}{2} \ln \frac{1+\sin x}{1-\sin x}.$$

Now multiply the numerator and the denominator by $1 - \sin x$:

$$\begin{aligned} \int \sec x dx &= \frac{1}{2} \ln \frac{1 - \sin^2 x}{(1 - \sin x)^2} \\ &= \frac{1}{2} \ln \frac{\cos^2 x}{(1 - \sin x)^2} = \ln \left| \frac{\cos x}{1 - \sin x} \right| \end{aligned}$$

13. a. Write the differential equation as $dP/dt = .2P(1 - P/10) = .02P(10 - P)$. Now separate variables:

$$.02 dt = \frac{dP}{P(10 - P)}, \quad .02t + k = \int \frac{dP}{P(10 - P)}.$$

We can find this integral by using partial fractions:

$$\begin{aligned} \int \frac{dP}{P(10 - P)} &= .1 \int \frac{dP}{P} + .1 \int \frac{dP}{10 - P} \\ &= .1 \ln|P| - .1 \ln|10 - P| = .1 \ln \left| \frac{P}{10 - P} \right|. \end{aligned}$$

Thus

$$.2t + 10k = \ln \left| \frac{P}{10 - P} \right|, \quad \frac{P}{10 - P} = Be^{.2t},$$

where $B = \pm e^{10k}$ is an arbitrary constant. Now solve for P :

$$\begin{aligned} P &= 10Be^{.2t} - Be^{.2t}P \\ P + Be^{.2t}P &= 10Be^{.2t} \\ P(t) &= \frac{10Be^{.2t}}{1 + Be^{.2t}} = \frac{10}{1 + Ce^{-.2t}}, \end{aligned}$$

where $C = 1/B$. The initial condition determines C : $.5 = P(0) = 10/(1 + C)$, so $1 + C = 20$, $C = 19$:

$$P(t) = \frac{10}{1 + 19e^{-.2t}}.$$

In this form it is clear that $P(t) \rightarrow 10$ as $t \rightarrow \infty$.

13. b. $P(3) = 10/(1 + 19e^{-.6}) = .875008$ lb; $P(10) = 10/(1 + 19e^{-.2}) = 2.80005$ lb.

13. c. We need to solve for t : $5 = P(t) = 10/(1 + 19e^{-.2t})$.

$$\begin{aligned} 5 &= P(t) = 10/(1 + 19e^{-.2t}) \\ 1 + 19e^{-.2t} &= 2, \quad e^{-.2t} = \frac{1}{19}, \end{aligned}$$

so $.2t = \ln(19)$, $t = 5 \ln(19) = 14.7222$ hours.

11.5 Trigonometric Integrals

1. In the following, we give a trigonometric substitution that will work, and then indicate the value of the integral as I .

1. a. $2x = \sin u; I = \frac{1}{2} \arcsin 2x$.

1. b. $2x = \tan u; I = \frac{1}{2} \ln |\sec u + \tan u|$
 $= \frac{1}{2} \ln |\sqrt{1+4x^2} + 2x|$.

1. c. $x = 2 \tan u; I = \ln |\sec u + \tan u|$
 $= \ln |\sqrt{4+x^2} + x|$ (or $\ln |\sqrt{4+x^2} + x| - \ln 2$).

1. d. Not a trig substitution: $u = 4 + x^2; I = \sqrt{4+x^2}$.

1. e. $x = a \sin u; I = \int \frac{a \cos u du}{a^3 \cos^3 u}$
 $= \frac{1}{a^2} \int \sec^2 u du = \frac{1}{a^2} \tan u = \frac{x}{a^2 \sqrt{a^2 - x^2}}$.

1. f. $x = 2 \tan u; I = \frac{1}{2} u = \frac{1}{2} \arctan(x/2)$.

1. g. Not a trig substitution: $u = 4 + x^2; I = \frac{1}{2} \ln u = \ln \sqrt{4+x^2}$.

1. h. $x = 2 \tan u; I = \int \frac{\sec^2 u du}{2 \tan u \sec u}$
 $= \frac{1}{2} \int \csc u du = -\frac{1}{2} \ln |\csc u + \cot u|$
 $= -\ln \sqrt{\left| \frac{\sqrt{4+x^2}}{x} + \frac{2}{x} \right|} = \ln \sqrt{\left| \frac{x}{\sqrt{4+x^2}} + \frac{x}{2} \right|}$

1. i. Not a trig substitution: $u = x^2 - a^2; I = \sqrt{u} = \sqrt{x^2 - a^2}$.

1. j. $x = a \tan u; I = \int \frac{a \sec^2 u du}{(a^2 \sec^2 u)^2}$
 $= \frac{1}{a^3} \int \cos^2 u du = \frac{2u + \sin 2u}{4a^3}$
 $= \frac{u + \sin u \cos u}{2a^3} = \frac{1}{2a^3} \left(\arctan(x/a) + \frac{ax}{(a^2 + x^2)} \right)$.

2. a. $x = 2 \sin u; -\pi/6 \leq u \leq \pi/6$ when $-1 \leq x \leq 1$, so

$$\begin{aligned} \int_1^{-1} \frac{dx}{4-x^2} &= \int_{\pi/6}^{-\pi/6} \frac{2 \cos u du}{4 \cos^2 u} \\ &= \frac{1}{2} \int_{\pi/6}^{-\pi/6} \sec u du = \ln \sqrt{|\sec u + \tan u|} \Big|_{\pi/6}^{-\pi/6} \\ &= \ln \sqrt{\frac{\sec(-\pi/6) + \tan(-\pi/6)}{\sec(\pi/6) + \tan(\pi/6)}} \\ &= \ln \sqrt{\frac{2/\sqrt{3} - 1/\sqrt{3}}{2/\sqrt{3} + 1/\sqrt{3}}} = \ln \sqrt{1/3} = -\ln \sqrt{3}. \end{aligned}$$

Note: the integral has a negative value because the domain of integration is traversed in the negative direction.

2. b. Let $x = \sec u; I = \int \tan^2 u \sec u du =$

$$\int (\sec^2 u - 1) \sec u du = \int \cos^{-3} u du - \int \sec u du.$$

The reduction formulas on page 745 of the text give

$$\int \cos^{-3} u du = \frac{1}{2} \cos^{-2} u \sin u + \frac{1}{2} \int \sec u du.$$

Thus $I = \frac{1}{2} \left(\cos^{-2} u \sin u - \int \sec u du \right)$
 $= \frac{1}{2} \left(\sec u \tan u - \int \sec u du \right).$

Since $\int \sec u du = \ln |\sec u + \tan u|$, we get finally

$$\begin{aligned} I &= \frac{\sec u \tan u - \ln |\sec u + \tan u|}{2} \\ &= \frac{x\sqrt{x^2-1} - \ln |x + \sqrt{x^2-1}|}{2} \end{aligned}$$

Therefore, $\int_1^2 \sqrt{x^2-1} dx =$

$$\left. \frac{x\sqrt{x^2-1} - \ln |x + \sqrt{x^2-1}|}{2} \right|_1^2 = \sqrt{3} - \ln \sqrt{2+\sqrt{3}}.$$

2. c. First use integration by parts: $F(x) = x, g(x) = \sec^2 x$:

$$\int x \sec^2 x dx = x \tan x - \int \tan x dx = x \tan x + \ln |\cos x|.$$

Therefore

$$\int_0^{\pi/3} x \sec^2 x dx = x \tan x + \ln |\cos x| \Big|_0^{\pi/3} = \frac{\pi\sqrt{3}}{3} - \ln 2$$

2. d. Use the solution to exercise 1e (with $a = \sqrt{2}$):

$$\int_0^1 \frac{dx}{(2-x^2)^{3/2}} = \frac{x}{2\sqrt{2-x^2}} \Big|_0^1 = \frac{1}{2}.$$

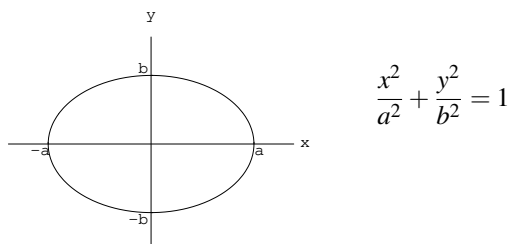
2. e. Modify the solution to exercise 1f:

$$\int_0^\infty \frac{dx}{9+x^2} = \frac{1}{3} \arctan(x/3) \Big|_0^\infty = \frac{1}{3} \cdot \frac{\pi}{2} = \frac{\pi}{6}.$$

2. f. No trig substitution here. Let $u = x^2 - a^2$. Then $du = 2x dx$, so $x^3 dx = x^2 \cdot x dx = (u + a^2) \cdot \frac{1}{2} du$. Also, $0 \leq u \leq 3a^2$ when $a \leq x \leq 2a$, so

$$\begin{aligned} \int_a^{2a} x^3 \sqrt{x^2 - a^2} dx &= \frac{1}{2} \int_0^{3a^2} u^{3/2} + a^2 u^{1/2} du \\ &= \left(\frac{u^{5/2}}{5} + \frac{a^2 u^{3/2}}{3} \right) \Big|_0^{3a^2} \\ &= \frac{3^{5/2}}{5} a^5 + \frac{3^{3/2}}{3} a^5 = \frac{14a^5 \sqrt{3}}{5}. \end{aligned}$$

3.



The total area of the ellipse is four times the area in the first quadrant. The part of the ellipse in the first quadrant is the graph of $y = \frac{b}{a} \sqrt{1 - x^2}$, where $0 \leq x \leq a$. Its area is

$$\frac{b}{a} \int_0^a \sqrt{a^2 - x^2} dx = ab \int_0^{\pi/2} \cos^2 u du$$

where $x = a \sin u$. From section 11.4 we find that the value of this expression is

$$ab \left(\frac{u}{2} + \frac{\sin 2u}{4} \right) \Big|_0^{\pi/2} = ab \cdot \frac{\pi}{4}$$

Since this was one-fourth the total area of the ellipse, we have shown that the area of the ellipse is πab .

4. Again, we let I denote the integral to be found.

4. a. $x^2 - 2x - 8 = (x - 1)^2 - 9$. Let $x - 1 = 3 \sec u$; then

$$I = \int \frac{\sec u \tan u dx}{\tan u} = \int \sec u du = \ln |\sec u + \tan u|.$$

Since $3 \tan u = \sqrt{(x - 1)^2 - 3^2} = \sqrt{x^2 - 2x - 8}$, we have

$$\begin{aligned} I &= \ln \left| \frac{x - 1}{3} + \frac{\sqrt{x^2 - 2x - 8}}{3} \right| + C \\ &= \ln |x - 1 + \sqrt{x^2 - 2x - 8}| + C'. \end{aligned}$$

In this solution we made the arbitrary constant C explicit so we could show how $-\ln 3$ got absorbed and made a new constant $C' = C - \ln 3$.

$$\mathbf{4. b.} \quad I = \int \frac{dx}{(x + 3)^2 + 1} =$$

$$\int \frac{du}{u^2 + 1} = \arctan u = \arctan(x + 3)$$

$$\mathbf{4. c.} \quad I = \int \frac{dx}{\sqrt{(x + 3)^2 - 1}} = \int \frac{du}{\sqrt{u^2 - 1}}. \quad \text{Now let } u = \sec w:$$

$$\begin{aligned} I &= \int \sec w dw = \ln |\sec w + \tan w| \\ &= \ln |u + \sqrt{u^2 - 1}| = \ln |x + 3 + \sqrt{x^2 + 6x + 8}|. \end{aligned}$$

$$\mathbf{4. d.} \quad I = \int \frac{x dx}{(x + 2)^2 + 1} = \int \frac{(u - 2) du}{u^2 + 1}$$

$$\begin{aligned} &= \frac{1}{2} \ln(u^2 + 1) - 2 \arctan u \\ &= \ln \sqrt{x^2 + 4x + 5} - 2 \arctan(x + 2). \end{aligned}$$

$$\mathbf{4. e.} \quad I = \int \frac{x dx}{\sqrt{9 - (x - 2)^2}} = \int \frac{(u + 2) du}{\sqrt{9 - u^2}}$$

$$\begin{aligned} &= -\sqrt{9 - u^2} + \frac{2}{3} \arcsin(u/3) \\ &= -\sqrt{5 + 4x - x^2} + \frac{2}{3} \arcsin \left(\frac{x - 2}{3} \right) \end{aligned}$$

$$\mathbf{4. f.} \quad I = \int \frac{2x + 1 + 6 dx}{(2x + 1)^2 + 4} = \frac{1}{2} \int \frac{u du}{u^2 + 4} + 6 \int \frac{du}{u^2 + 4}$$

$$\begin{aligned} &= \frac{1}{4} \ln(u^2 + 4) + 3 \arctan(u/2) \\ &= \frac{1}{4} \ln(4x^2 + 4x + 5) + \arctan \left(\frac{2x + 1}{2} \right) \end{aligned}$$

$$\mathbf{4. g.} \quad I = \int \frac{4(x + 1) - 7 dx}{\sqrt{1 - (x + 1)^2}} =$$

$$\begin{aligned} &4 \int \frac{u du}{\sqrt{1 - u^2}} - 7 \int \frac{du}{\sqrt{1 - u^2}} = -4 \sqrt{1 - u^2} - 7 \arcsin u \\ &= -4 \sqrt{-2x - x^2} - 7 \arcsin(x + 1) \end{aligned}$$

$$\mathbf{4. h.} \quad I = \int \frac{dx}{(a^2 + 1 - (x + 1)^2)^2} = \int \frac{dy}{(b^2 - y^2)^2},$$

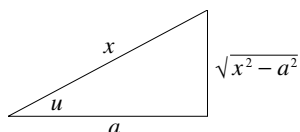
where $b^2 = a^2 + 1$. Now set $y = b \sin u$;

$$I = \int \frac{b \cos u du}{b^4 \cos^4 u} = \frac{1}{b^3} \int \cos^{-3} u du.$$

The reduction formulas on page 745 of the text then allow us to write

$$\begin{aligned}
 I &= \frac{1}{b^3} \left(\frac{1}{2} \cos^{-2} u \sin u + \frac{1}{2} \int \cos^{-1} u du \right) \\
 &= \frac{1}{2b^3} \left(\frac{\sin u}{\cos^2 u} + \ln |\sec u + \tan u| \right) \\
 &= \frac{1}{2b^3} \left(\frac{by}{b^2 - y^2} + \ln \left| \frac{b}{\sqrt{b^2 - y^2}} + \frac{y}{\sqrt{b^2 - y^2}} \right| \right) \\
 &= \frac{(x+1)}{2(a^2+1)(a^2-x^2-2x)} \\
 &\quad + \frac{1}{2(a^2+1)^{3/2}} \ln \left| \frac{x+1+\sqrt{a^2+1}}{\sqrt{a^2-x^2-2x}} \right|
 \end{aligned}$$

5. a.



5. b. $\sin u = \frac{\sqrt{x^2 - a^2}}{x}$.

5. c. $\cos u = \frac{a}{x}$. d. $\tan u = \frac{\sqrt{x^2 - a^2}}{a}$.

6. Let I denote the integral being sought.

6. a. $\frac{1}{\sin x \cos x} = \frac{\cos x}{\sin x \cos^2 x} = \frac{\sec^2 x}{\tan x}$. Therefore

$$I = \int \frac{\sec^2 x}{\tan x} dx = \int \frac{du}{u} = \ln |u| = \ln |\tan x|.$$

6. b. $\cos^3 x \sin^{-4} x = \cos x (1 - \sin^2 x) \sin^{-4} x$
 $= \cos x (\sin^{-4} x - \sin^{-2} x).$

Therefore $I = \int \cos x (\sin^{-4} x - \sin^{-2} x) dx$
 $= \int (u^{-4} - u^{-2}) du = \frac{u^{-3}}{-3} - \frac{u^{-1}}{-1} = \frac{1}{\sin x} - \frac{1}{3 \sin^3 x}.$

6. c. $\csc^4 x \cot^2 x = \sin^{-4} x \cos^2 x \sin^{-2} x = \sin^{-6} x (1 - \sin^2 x) = \sin^{-6} x - \sin^{-4} x$. Now consider the reduction formulas:

$$\begin{aligned}
 \int \sin^{-2} x dx &= -\sin^{-1} x \cos x = -\frac{\cos x}{\sin x} \\
 \int \sin^{-4} x dx &= -\frac{1}{3} \sin^{-3} x \cos x + \frac{2}{3} \int \sin^{-2} x dx \\
 &= -\frac{\cos x}{3 \sin^3 x} - \frac{2 \cos x}{3 \sin x} \\
 \int \sin^{-6} x dx &= -\frac{1}{5} \sin^{-5} x \cos x + \frac{4}{5} \int \sin^{-4} x dx \\
 &= -\frac{\cos x}{5 \sin^5 x} - \frac{4 \cos x}{15 \sin^3 x} - \frac{8 \cos x}{15 \sin x}
 \end{aligned}$$

It follows that

$$\begin{aligned}
 I &= -\frac{\cos x}{5 \sin^5 x} - \frac{4 \cos x}{15 \sin^3 x} - \frac{8 \cos x}{15 \sin x} \\
 &\quad + \frac{\cos x}{3 \sin^3 x} + \frac{2 \cos x}{3 \sin x} \\
 &= -\frac{\cos x}{5 \sin^5 x} + \frac{\cos x}{15 \sin^3 x} + \frac{2 \cos x}{15 \sin x} \\
 &= -\frac{1}{5} \csc^4 x \cot x + \frac{1}{15} \csc^2 x \cot x + \frac{2}{15} \cot x.
 \end{aligned}$$

6. d. $\sin 3x \cot 3x = \sin 3x \frac{\cos 3x}{\sin 3x} = \cos 3x$, so

$$I = \int \cos 3x dx = \frac{1}{3} \sin 3x.$$

6. e. $\sin^n x \cos^3 x = \sin^n x \cos^2 x \cos x$
 $= \sin^n x (1 - \sin^2 x) \cos x = (\sin^n x - \sin^{n+2} x) \cos x$. Therefore,
 $I = \int_0^{\pi/2} (\sin^n x - \sin^{n+2} x) \cos x dx$

$$\begin{aligned}
 &= \int_0^1 u^n - u^{n+2} du = \left. \frac{u^{n+1}}{n+1} - \frac{u^{n+3}}{n+3} \right|_0^1 \\
 &= u^{n+1} \left(\frac{1}{n+1} - \frac{u^2}{n+3} \right) \Big|_0^1 \\
 &= \begin{cases} \frac{2}{(n+1)(n+3)} & n > -1 \\ \infty & n \leq -1. \end{cases}
 \end{aligned}$$

6. f. $\tan^5 x = \sin^5 x \cos^{-5} x = \sin x (\sin^2 x)^2 \cos^{-5} x = \sin x (1 - \cos^2 x)^2 \cos^{-5} x = \sin x (1 - 2 \cos^2 x + \cos^4 x) \cos^{-5} x = \sin x (\cos^{-5} x - 2 \cos^{-3} x + \cos^{-1} x)$. Therefore,

$$\begin{aligned}
 I &= \int (\cos^{-5} x - 2 \cos^{-3} x + \cos^{-1} x) \sin x dx \\
 &= \int u^{-5} - 2u^{-3} + u^{-1} du = \frac{u^{-4}}{-4} + \frac{u^{-2}}{-2} + \ln |u| \\
 &= \ln |\cos x| - \frac{1}{2 \cos^2 x} - \frac{1}{4 \cos^4 x}.
 \end{aligned}$$

6. g. $\frac{\sin^3 5x}{\sqrt[3]{\cos 5x}} = \sin 5x \sin^2 5x \cos^{-1/3} 5x$
 $= \sin 5x (1 - \cos^2 5x) \cos^{-1/3} 5x$
 $= \sin 5x (\cos^{-1/3} 5x - \cos^{5/3} 5x)$. Therefore

$$\begin{aligned}
 I &= \int (\cos^{-1/3} 5x - \cos^{5/3} 5x) \sin 5x dx \\
 &= -\frac{1}{5} \int u^{-1/3} - u^{5/3} du = -\frac{1}{5} \left(\frac{3u^{2/3}}{2} - \frac{3u^{8/3}}{8} \right) \\
 &= \frac{3}{10} \left(\frac{\cos^{8/3} 5x}{4} - \cos^{2/3} 5x \right).
 \end{aligned}$$

6. h. Let $u = \ln x$; then $du = dx/x$ and

$$\begin{aligned} I &= \int \cos^3 u du = \frac{1}{3} \cos^2 u \sin u + \frac{2}{3} \int \cos u du \\ &= \frac{1}{3} \cos^2 u \sin u + \frac{2}{3} \sin u = \frac{\sin(\ln x)}{3} (\cos^2(\ln x) + 2). \end{aligned}$$

In the first line we used the reduction formulas on page 745.

6. i. Let $u = \tan x$; then $du = \sec^2 x dx$ and

$$\begin{aligned} I &= \int \sec^2 x \ln(\tan x) \sec^2 x dx \\ &= \int (u^2 + 1) \ln u du = \int u^2 \ln u du + \int \ln u du. \end{aligned}$$

The second integral is example 2 in the text, page 710, $\int \ln u du = u \ln u - u$. Use integration by parts in the first integral: $F(u) = u^2$, $g(u) = \ln u$, so $G(u) = u \ln u - u$ and

$$\begin{aligned} J &= \int u^2 \ln u du \\ &= u^2(u \ln u - u) - 2 \int u^2 \ln u du + 2 \int u^2 du \\ &= u^3 \ln u - u^3 - 2J + \frac{2u^3}{3}. \end{aligned}$$

Therefore $3J = u^3(3 \ln u - 1)/3$, so

$J = u^3(3 \ln u - 1)/9$ and

$$\begin{aligned} I &= u^3 \frac{3 \ln u - 1}{9} + u(\ln u - 1) \\ &= \tan^3 x \frac{3 \ln(\tan x) - 1}{9} + \tan x (\ln(\tan x) - 1). \end{aligned}$$

6. j. From the solution to exercise 1e,

$$I = \frac{x}{a^2 \sqrt{a^2 - x^2}} \Big|_0^{a/2} = \frac{a/2}{a^2 \sqrt{a^2 - a^2/4}} = \frac{1}{a^2 \sqrt{3}}.$$

We assume here that $a > 0$; otherwise $a/\sqrt{a^2} = -1$ and $I = -1/2a^2\sqrt{3}$.

7. We need the following integrals, which we obtain using the reduction formulas on page 745:

$$\begin{aligned} I_1 &= \int \cos^{-1} x dx = \ln |\sec x + \tan x| \\ I_3 &= \int \cos^{-3} x dx = \frac{1}{2} \cos^{-2} x \sin x + \frac{1}{2} \int \cos^{-1} x dx \\ &= \frac{\sin x}{2 \cos^2 x} + \frac{1}{2} \ln |\sec x + \tan x| \\ I_5 &= \int \cos^{-5} x dx = \frac{1}{4} \cos^{-4} x \sin x + \frac{3}{4} \int \cos^{-3} x dx \\ &= \frac{\sin x}{4 \cos^4 x} + \frac{3 \sin x}{8 \cos^2 x} + \frac{3}{8} \ln |\sec x + \tan x| \end{aligned}$$

Then, using the formula on page 746, we can say

$$\begin{aligned} I &= \int \sin^{-4} x \cos^{-5} x dx \\ &= \int (\sin^{-4} x + 3 \sin^{-2} x) \cos x dx + 3I_1 + 2I_3 + I_5. \end{aligned}$$

The first of these integrals can be found with the substitution $u = \sin x$:

$$\int u^{-4} + 3u^{-2} du = \frac{u^{-3}}{-3} + 3 \frac{u^{-1}}{-1} = -\frac{1}{3 \sin^3 x} - \frac{3}{\sin x}$$

Now put all the pieces together:

$$\begin{aligned} I &= \frac{\sin x}{4 \cos^4 x} + \frac{11 \sin x}{8 \cos^2 x} + \frac{35}{8} \ln |\sec x + \tan x| \\ &\quad - \frac{1}{3 \sin^3 x} - \frac{3}{\sin x} \end{aligned}$$

Reduction Formulas

8. Use integration by parts on $I = \int \cos^m x dx$. Let $F(x) = \cos^{m-1} x$, $g(x) = \cos x$. Then $G(x) = \sin x$ and $f(x) = -(m-1) \cos^{m-2} x \sin x$, so

$$\begin{aligned} I &= \cos^{m-1} x \sin x + (m-1) \int \cos^{m-2} x \sin^2 x dx \\ &= \cos^{m-1} x \sin x + (m-1) \int \cos^{m-2} x (1 - \cos^2 x) dx \\ &= \cos^{m-1} x \sin x + (m-1) \int \cos^{m-2} x dx - (m-1)I. \end{aligned}$$

Collect on the left all appearances of I ,

$$mI = \cos^{m-1} x \sin x + (m-1) \int \cos^{m-2} x dx,$$

and divide by m to get the reduction formula

$$I = \frac{1}{m} \cos^{m-1} x \sin x + \frac{m-1}{m} \int \cos^{m-2} x dx.$$

To get the other recursion formula (for negative exponents), rewrite the first recursion relation so that $\int \cos^{m-2} x dx$ is isolated on the left-hand side:

$$\begin{aligned} \int \cos^{m-2} x dx &= \\ &= -\frac{m}{m-1} \frac{1}{m} \cos^{m-1} x \sin x + \frac{m}{m-1} \int \cos^m x dx \\ &= \frac{1}{1-m} \cos^{m-1} x \sin x + \frac{m}{m-1} \int \cos^m x dx \end{aligned}$$

Now call $m-2$ by a new name $-k$:

$$\begin{aligned} \int \cos^{-k} x dx &= \\ &= \frac{1}{k-1} \cos^{1-k} x \sin x + \frac{k-2}{k-1} \int \cos^{-k+2} x dx \end{aligned}$$

This is the reduction formula for negative exponents.

9. a. Since $\tan^n x = \tan^{n-2} x (\sec^2 x - 1)$,

$$\begin{aligned}\int \tan^n x dx &= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx \\ &= \int u^{n-2} du - \int \tan^{n-2} x dx \\ &= \frac{1}{n-1} u^{n-1} - \int \tan^{n-2} x dx \\ &= \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x dx.\end{aligned}$$

9. b. We establish the recursions “from the bottom up”:

$$\begin{aligned}\int \tan^2 x dx &= \tan x - x \\ \int \tan^4 x dx &= \frac{1}{3} \tan^3 x - \int \tan^2 x dx \\ &= \frac{1}{3} \tan^3 x - \tan x + x \\ \int \tan^6 x dx &= \frac{1}{5} \tan^5 x - \int \tan^4 x dx \\ &= \frac{1}{5} \tan^5 x - \frac{1}{3} \tan^3 x + \tan x - x\end{aligned}$$

9. c. We see the pattern for even n in the last part:

$$\begin{aligned}\int \tan^{2k} x dx &= \frac{\tan^{2k-1}}{2k-1} - \frac{\tan^{2k-3}}{2k-3} + \cdots \\ &\quad + (-1)^k \tan x - (-1)^k x.\end{aligned}$$

The pattern for n odd starts with $n = 3$:

$$\begin{aligned}\int \tan^3 x dx &= \frac{\tan^2 x}{2} - \int \tan x dx \\ &= \frac{\tan^2 x}{2} + \ln |\cos x|.\end{aligned}$$

Therefore,

$$\begin{aligned}\int \tan^{2k+1} x dx &= \frac{\tan^{2k}}{2k} - \frac{\tan^{2k-2}}{2k-2} + \cdots \\ &\quad + (-1)^{k+1} \tan^2 x + (-1)^{k+1} \ln |\cos x|.\end{aligned}$$

The value of $\int_0^{\pi/4} \tan^n x dx$ is obtained by evaluating the right-hand side at $x = \pi/4$. (The value at $x = 0$ is zero.) When $n = 2k$, we get

$$\frac{1}{2k-1} - \frac{1}{2k-3} + \cdots + (-1)^k \cdot 1 - (-1)^k \frac{\pi}{4}.$$

To do this when n is odd, note that $\cos(\pi/4) = 1/\sqrt{2}$, so $\ln |\cos x| = \ln(1/\sqrt{2}) = -\ln(\sqrt{2}) = -\frac{1}{2} \ln 2$. Therefore, when $n = 2k+1$, we get

$$\frac{1}{2k} - \frac{1}{2k-2} + \cdots + (-1)^{k+1} \frac{1}{2} - (-1)^{k+1} \frac{1}{2} \ln 2.$$

This agrees with the expressions in the statement of the exercise.

9. d. For every $n > 0$, $0 \leq \tan^n x \leq 1$ when $0 \leq x \leq \pi/4$. Therefore,

$$0 \leq \int_0^{\pi/4} \tan^n x dx \leq 1 \cdot \frac{\pi}{4}.$$

But also,

$$\begin{aligned}0 \leq \int_0^{\pi/4} \tan^n x dx &= \frac{1}{n-1} - \int_0^{\pi/4} \tan^{n-2} x dx \\ &\leq \frac{1}{n-1}.\end{aligned}$$

The last inequality holds because the second term (with its minus sign included) is negative. Thus

$$0 \leq \lim_{n \rightarrow \infty} \int_0^{\pi/4} \tan^n x dx \leq \lim_{n \rightarrow \infty} \frac{1}{n-1} = 0,$$

forcing $\lim_{n \rightarrow \infty} \int_0^{\pi/4} \tan^n x dx = 0$.

9. e. We can rewrite the formula for even n , obtained in part c, above, as

$$\begin{aligned}(-1)^k \frac{\pi}{4} &= \frac{1}{2k-1} - \frac{1}{2k-3} + \cdots + (-1)^k \cdot 1 \\ &\quad - \int_0^{\pi/4} \tan^{2k} x dx \\ \frac{\pi}{4} &= 1 - \frac{1}{3} + \cdots \pm \frac{1}{2k-1} \mp \int_0^{\pi/4} \tan^{2k} x dx.\end{aligned}$$

Now take the limit of this last equation as $n \rightarrow \infty$:

$$\frac{\pi}{4} = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{3} + \cdots \pm \frac{1}{2k-1} \right) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1},$$

which is true because we have just shown that the limit of the integral expression is 0.

9. f. We can rewrite the formula for odd n from part c, above, as

$$\begin{aligned}(-1)^{k+1} \frac{\ln 2}{2} &= \frac{1}{2k} - \frac{1}{2k-2} + \cdots + (-1)^{k+1} \cdot \frac{1}{2} \\ &\quad - \int_0^{\pi/4} \tan^{2k} x dx \\ \ln 2 &= 2 \left(\frac{1}{2} - \frac{1}{4} + \cdots \pm \frac{1}{2k} \mp \int_0^{\pi/4} \tan^{2k+1} x dx \right) \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \cdots \pm \frac{1}{k} \mp \frac{1}{2} \int_0^{\pi/4} \tan^{2k+1} x dx\end{aligned}$$

In the limit as $n \rightarrow \infty$, the integral disappears (by d) and we have

$$\ln 2 = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots \pm \frac{1}{k} \right) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}.$$

10. a. Let $F(x) = \sec^{n-2} x$, $g(x) = \sec^2 x$. Then $G(x) = \tan x$ and $f(x) = (n-2)\sec^{n-3} x \sec x \tan x$, so

$$\begin{aligned} I &= \int \sec^{n-2} x \sec^2 x dx \\ &= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x dx \\ &= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) dx \\ &= \sec^{n-2} x \tan x - (n-2)I + (n-2) \int \sec^{n-2} x dx. \end{aligned}$$

Therefore,

$$(n-1)I = \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x dx,$$

which becomes

$$I = \int \sec^n x dx = \frac{1}{n-1} \sec^{n-2} x \tan x + \frac{n-2}{n-1} \int \sec^{n-2} x dx.$$

This is the recursion relation.

10. b. The second cosine recursion formula, page 745, becomes

$$\begin{aligned} \int \sec^m x dx &= \frac{1}{m-1} \sec^{m-1} x \sin x + \frac{m-2}{m-1} \int \sec^{m-2} x dx \\ &= \frac{1}{m-1} \sec^{m-2} x \frac{\sin x}{\cos x} + \frac{m-2}{m-1} \int \sec^{m-2} x dx \\ &= \frac{1}{m-1} \sec^{m-2} x \tan x + \frac{m-2}{m-1} \int \sec^{m-2} x dx. \end{aligned}$$

This is the same as the equation in part a.

11. a. Use integration by parts on $\int x^n e^x dx$. Let $F(x) = x^n$, $g(x) = e^x$; then

$$\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx.$$

11. b. Let $a_n = \int_0^t x^n e^x dx$. Then part a says

$$a_n = x^n e^x \Big|_0^t - n a_{n-1} = t^n e^t - n a_{n-1}$$

if n is a positive integer, while $a_0 = e^t - 1$. If n is a positive integer, we can rewrite the expression as

$$\frac{a_n}{n!} = \frac{t^n e^t}{n!} - \frac{a_{n-1}}{(n-1)!}.$$

$$\text{Hence } \frac{a_n}{n!} = \frac{t^n e^t}{n!} - \frac{a_{n-1}}{(n-1)!} =$$

$$\begin{aligned} &= \frac{t^n e^t}{n!} - \frac{t^{n-1} e^t}{(n-1)!} + \frac{a_{n-2}}{(n-2)!} \\ &= \frac{t^n e^t}{n!} - \frac{t^{n-1} e^t}{(n-1)!} + \frac{t^{n-2} e^t}{(n-2)!} - \frac{a_{n-3}}{(n-3)!} \\ &\vdots \\ &= \frac{t^n e^t}{n!} - \frac{t^{n-1} e^t}{(n-1)!} + \frac{t^{n-2} e^t}{(n-2)!} - \cdots \pm \frac{a_2}{2!} \\ &= \frac{t^n e^t}{n!} - \frac{t^{n-1} e^t}{(n-1)!} + \frac{t^{n-2} e^t}{(n-2)!} - \cdots \pm \frac{t^2 e^t}{2!} \mp \frac{a_1}{1!} \\ &= \frac{t^n e^t}{n!} - \frac{t^{n-1} e^t}{(n-1)!} + \frac{t^{n-2} e^t}{(n-2)!} - \cdots \pm \frac{t^2 e^t}{2!} \mp \frac{t e^t}{1!} \pm a_0 \\ &= \frac{t^n e^t}{n!} - \frac{t^{n-1} e^t}{(n-1)!} + \frac{t^{n-2} e^t}{(n-2)!} - \cdots \\ &\quad \cdots \pm \frac{t^2 e^t}{2!} \mp \frac{t e^t}{1!} \pm (e^t - 1). \end{aligned}$$

Now factor out e^t ; the result is exactly the expression we seek:

$$\begin{aligned} \frac{1}{n!} \int_0^t x^n e^x dx &= e^t \left(\frac{t^n}{n!} - \frac{t^{n-1}}{(n-1)!} + \frac{t^{n-2}}{(n-2)!} - \cdots \right. \\ &\quad \left. \cdots \pm \frac{t^2}{2!} \mp \frac{t}{1!} \pm 1 \right) \mp 1. \end{aligned}$$

11. c. Use the same approach as in exercise 9d: We can convert the basic recursion relation into a nest of inequalities of the form

$$0 \leq \frac{a_n}{n!} = \frac{t^n e^t}{n!} - \frac{a_{n-1}}{(n-1)!} \leq \frac{t^n e^t}{n!} = \frac{K t^n}{n!}.$$

This is possible because each $a_k/k!$ is non-negative, so $-a_{n-1}/(n-1)! \leq 0$. Also, to emphasize that t is fixed, we write e^t as a constant K . We now show the right-hand side goes to zero as $n \rightarrow \infty$. Stirling's formula helps here: $n! \sim n^n \sqrt{2\pi n} e^{-n}$ (text page 768).

$$\frac{K t^n}{n!} \sim \frac{K (et)^n}{n^n \sqrt{2\pi n}} = \frac{K}{\sqrt{2\pi n}} \left(\frac{et}{n} \right)^n \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Thus

$$0 \leq \lim_{n \rightarrow \infty} \frac{a_n}{n!} \leq \lim_{n \rightarrow \infty} \frac{t^n e^t}{n!} = 0,$$

forcing $\lim_{n \rightarrow \infty} \frac{a_n}{n!} = \lim_{n \rightarrow \infty} \frac{1}{n!} \int_0^t x^n e^x dx = 0$.

11. d. First rewrite the expression we got in part b in the form

$$e^{-t} = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots \pm \frac{t^n}{n!} \mp e^{-t} \cdot \frac{1}{n!} \int_0^t x^n e^x dx.$$

Since the integral vanishes in the limit as $n \rightarrow \infty$ we find

$$e^{-t} = \lim_{n \rightarrow \infty} \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots \pm \frac{t^n}{n!} \right).$$

12. a. Trig substitution: $x = \tan u$. Then $1 + x^2 = \sec^2 u$ and

$$\int \frac{dx}{(1+x^2)^n} = \int \frac{\sec^2 u du}{\sec^{2n} u} = \int \cos^{2n-2} u du.$$

Now use the first cosine reduction formula (page 745):

$$\begin{aligned} \int \cos^{2n-2} u du &= \frac{1}{2n-2} \cos^{2n-3} u \sin u \\ &\quad + \frac{2n-3}{2n-2} \int \cos^{2n-4} u du. \end{aligned}$$

The new integral is

$$\int \sec^{4-2n} u du = \int \frac{\sec^2 u du}{\sec^{2n-2} u} = \int \frac{dx}{(1+x^2)^{n-1}}.$$

Since $\sin u = x/\sqrt{1+x^2}$ and $\cos u = 1/\sqrt{1+x^2}$, the complete recursion relation takes the form

$$\begin{aligned} \int \frac{dx}{(1+x^2)^n} &= \\ &= \frac{1}{(2n-2)} \frac{x}{(1+x^2)^{n-1}} + \frac{2n-3}{2n-2} \int \frac{dx}{(1+x^2)^{n-1}}. \end{aligned}$$

12. b. To find the exact value of

$$I_n = \int_0^1 \frac{dx}{(1+x^2)^n}$$

for $n = 5$, we compute recursively:

$$\begin{aligned} I_1 &= \arctan x \Big|_0^1 = \frac{\pi}{4} \\ I_2 &= \frac{1}{2} \frac{x}{1+x^2} \Big|_0^1 + \frac{1}{2} I_1 = \frac{2+\pi}{2^3} \\ I_3 &= \frac{1}{4} \frac{x}{(1+x^2)^2} \Big|_0^1 + \frac{3}{4} I_2 = \frac{8+2\pi}{2^5} \\ I_4 &= \frac{1}{6} \frac{x}{(1+x^2)^3} \Big|_0^1 + \frac{5}{6} I_3 = \frac{44+15\pi}{3 \cdot 2^6} \\ I_5 &= \frac{1}{8} \frac{x}{(1+x^2)^4} \Big|_0^1 + \frac{7}{8} I_4 = \frac{320+105\pi}{3 \cdot 2^9}, \end{aligned}$$

and we can take the last as the exact value of I_5 . The approximate value is $I_5 \approx 0.42309064363732310549$.

13. *Ad libitum*. This is a long project.

11.6 Simpson's Rule

1. In the table, S is the value given by Simpson's rule, and the underlined digits agree with those of π :

n	S
2	<u>3.141</u> 568627...
4	<u>3.141592</u> 502458...
8	<u>3.141592651</u> 224...
16	<u>3.141592653552</u> ...

2. a. (To show $|f^{(4)}(x)| \leq 96$, calculate the fourth derivative, plot it using a computer, and then look for the largest absolute value.)

Use the bound given on page 755 of the text, with $M = 96$:

$$\left| \int_0^1 \frac{4dx}{1+x^2} - \text{Simpson's rule} \right| \leq \frac{96}{2880n^4} = \frac{1}{30n^4}.$$

Simpson's rule gives the value of the integral to 10 decimal places if the expression on the left is less than $.5 \times 10^{-10}$. This will be guaranteed if $1/30n^4$ is less than this value. We are given that $n = 256 = 2^8$, so

$$\frac{1}{30n^4} = \frac{1}{30 \times 2^{32}} \approx 7.76 \times 10^{-12} < .5 \times 10^{-10}.$$

2. b. If $n = 2^7$, then

$$\frac{1}{30n^4} = \frac{1}{30 \times 2^{28}} \approx 1.24 \times 10^{-10},$$

but this is *not* less than $.5 \times 10^{-10}$, so the error bound does not guarantee that Simpson's rule will give 10 decimal places correctly.

2. c. QuickBasic gives $S = 3.141592653589794$ when $n = 128$; the first 14 decimal places (i.e., all digits but the last) are correct. In fact, we saw in part a that $n = 16$ is sufficient to give the first 10 decimal places.

3. a. $n = 4$ is sufficient:

n	S
1	.747180...
2	.746855...
4	.746826...

3. b. $n = 64$ is sufficient:

n	S
1	6.235238728...
2	6.230303813...
4	6.229977567...
8	6.229960433...
16	6.229959451...
32	6.229959391...
64	6.229959388...

11.7 Improper Integrals

$$1. \text{ a. } \int_{-\infty}^0 e^x dx = \lim_{b \rightarrow -\infty} e^x \Big|_b^0 = \lim_{b \rightarrow -\infty} 1 - e^b = 1.$$

$$1. \text{ b. } \int_1^{\infty} \frac{du}{u} = \lim_{b \rightarrow \infty} \ln b - \ln 1 = +\infty.$$

$$1. \text{ c. } \int_0^1 \frac{dy}{y^2} = \lim_{b \rightarrow 0} \frac{-1}{y} \Big|_b^1 = \lim_{b \rightarrow 0} -1 + \frac{1}{b} = +\infty.$$

$$1. \text{ d. } \int_0^{\pi/2} \tan x dx = \lim_{b \rightarrow \pi/2} -\ln |\cos x| \Big|_0^b \\ = \lim_{b \rightarrow \pi/2} -\ln(\cos b) + \ln 1 = \lim_{\beta \rightarrow 0} -\ln \beta = +\infty.$$

$$1. \text{ e. } \int_0^{\infty} x e^{-x} dx = \lim_{b \rightarrow \infty} -(1+x)e^{-x} \Big|_0^b \\ = \lim_{b \rightarrow \infty} 1 - \frac{0(1+b)}{e^b} = 1.$$

Here we use the fact that exponential functions grow faster than powers: $x^k/e^x \rightarrow 0$ as $x \rightarrow \infty$ for any power k .

$$1. \text{ f. } \int_1^{\infty} \frac{du}{u^2} = \lim_{b \rightarrow \infty} \frac{-1}{u} = \lim_{b \rightarrow \infty} 1 - \frac{1}{b} = 1.$$

$$1. \text{ g. } \int_0^{\infty} \frac{x dx}{1+x^2} = \lim_{b \rightarrow \infty} \ln \sqrt{1+x^2} \Big|_0^b \\ = \lim_{b \rightarrow \infty} \ln \sqrt{1+b^2} = +\infty.$$

$$1. \text{ h. } \int_1^3 \frac{x dx}{x^2-1} = \lim_{b \rightarrow 1} \ln \sqrt{x^2-1} \Big|_b^3 \\ = \ln \sqrt{8} - \lim_{b \rightarrow 1} \ln \sqrt{b^2-1} = \ln \sqrt{8} - \lim_{\beta \rightarrow 0} \ln \beta = +\infty.$$

2. Let $J_n = \int_0^{\infty} dx/(1+x^2)^n$. The recursion relation from exercise 12, section 11.6 becomes

$$J_n = \lim_{b \rightarrow \infty} \frac{1}{2n-2} \frac{b}{(1+b^2)^{n-1}} + \frac{2n-3}{2n-2} J_{n-1}.$$

For every $n-1 \geq 1$, the first term is zero, so $J_n = (2n-3)J_{n-1}/(2n-2)$, a much simpler form. Thus

$$J_1 = \lim_{b \rightarrow \infty} \arctan x \Big|_0^b = \lim_{b \rightarrow \infty} \arctan b = \frac{\pi}{2}.$$

We can now compute the rest of the J_n with relative ease:

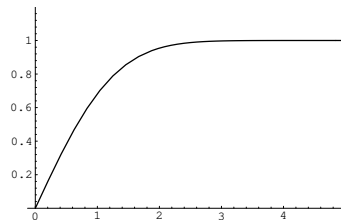
$$J_{10} = \frac{17}{18} J_9 = \frac{17}{18} \frac{15}{16} J_8 = \frac{17}{18} \frac{15}{16} \frac{13}{14} J_7 = \cdots \\ = \frac{17}{18} \frac{15}{16} \frac{13}{14} \frac{11}{12} \frac{9}{10} \frac{7}{8} \frac{5}{6} \frac{3}{4} \frac{1}{2} J_1 = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17}{2^{10}} \frac{\pi}{2}$$

3. When $b = 1$, RIEMANN gives the value .682689. When $b = 10$, 100, or 1000, the values RIEMANN gives round to 1.000000. This suggests

$$\lim_{b \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{-b}^b e^{-x^2/2} dx = 1.$$

4. a. The graph of $y = e^{-x^2/2}$ is symmetric about the y -axis, so the integral over the interval $[-b, b]$ will be twice the integral over $[0, b]$.

4. b.



The graph of $F(t)$, plotted above, suggests that $F(\infty) = 1$

4. c. The two results agree, and they agree with what the text claims for the value of this integral on page 759.

$$5. \Gamma(1) = \int_0^{\infty} e^{-t} dt = \lim_{b \rightarrow \infty} -e^{-t} \Big|_0^b = \lim_{b \rightarrow \infty} 1 - e^{-b} = 1.$$

6. Let $F(t) = t^x$, $g(t) = e^{-t}$. Then

$$\Gamma(x+1) = \int_0^{\infty} e^{-t} t^x dt \\ = -e^{-t} t^x \Big|_0^{\infty} + x \int_0^{\infty} e^{-t} t^{x-1} dt = x\Gamma(x),$$

because $t^n/e^t \rightarrow 0$ as $t \rightarrow \infty$.

7. For a positive integer n , $\Gamma(n) = (n-1)!$:

$$\begin{aligned} \Gamma(2) &= 1 \cdot \Gamma(1) = 1 & \Gamma(3) &= 2 \cdot \Gamma(2) = 2 \\ \Gamma(4) &= 3 \cdot \Gamma(3) = 3! & \Gamma(5) &= 4 \cdot \Gamma(4) = 4! \\ \Gamma(6) &= 5 \cdot \Gamma(5) = 5! \end{aligned}$$

If we let $n = \frac{3}{2}$, then we can write $\Gamma(\frac{3}{2}) = \frac{1}{2}!$, defining $\frac{1}{2}!$ in this way.

8. A numerical integrator gives these values for $S = \int_0^b e^{-t} t^{-1/2} dt$:

b	S
1	1.49365
5	1.76968
10	1.77244
100	1.77245
1000	1.77245

9. a. If $u = (2t)^{1/2}$, then $du = (2t)^{-1/2} dt$ (so $t^{-1/2} dt = \sqrt{2} du$), $t = u^2/2$, and $t^{-1/2} = 2^{1/2} u^{-1}$. Also, $u = 0$ when $t = 0$ and $u = \infty$ when $t = \infty$. Therefore

$$\Gamma(1/2) = \sqrt{2} \int_0^\infty e^{-u^2/2} du.$$

9. b. Exercise 4 gives

$$1 = \frac{\sqrt{2}}{\sqrt{\pi}} \int_0^\infty e^{-u^2/2} du = \frac{\Gamma(1/2)}{\sqrt{\pi}},$$

so $\Gamma(1/2) = \sqrt{\pi}$.

10. a. The recursion formula for the gamma function yields

$$\begin{aligned}\Gamma(3/2) &= \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2} \\ \Gamma(5/2) &= \frac{3}{2}\Gamma(3/2) = \frac{3\sqrt{\pi}}{4}.\end{aligned}$$

10. b. In exercise 7 we defined $\frac{1}{2}! = \Gamma(3/2)$, so we can now write $\frac{1}{2}! = \sqrt{\pi}/2$.

Solutions: Chapter 12

Case Studies

12.1 Stirling's Formula

1. *Ad libitum*. The answer depends on choices made by the student.

2. In more detail the values of Stirling's estimates are $9.324847625 \times 10^{157}$ and $4.0235372904 \times 10^{2567}$.

3. a.
$$a_k = \frac{N^k}{k!} = \frac{N}{k} \frac{N^{k-1}}{(k-1)!} = \frac{N}{k} a_{k-1}.$$

Therefore, if $k < N$, then $1 < N/k$ so $a_{k-1} < \frac{N}{k} a_{k-1} = a_k$.

If $k > N$, then $1 > N/k$ so $a_{k-1} > \frac{N}{k} a_{k-1} = a_k$.

If $k = N$, then $a_{k-1} = \frac{N}{k} a_{k-1} = a_k$.

3. b. If $k > 2N$, then $.5 = 1/2 > N/k$ so $.5a_{k-1} > \frac{N}{k} a_{k-1} = a_k$.

The inequality $a_{k+1} < \frac{1}{2} a_k$ holds for all $k > 2N$, so we have $a_{k+2} < \frac{1}{2} a_{k+1} < \frac{1}{2^2} a_k$, and so forth. Thus, more generally, $a_{k+m} < \frac{1}{2^m} a_k \rightarrow 0$ as $m \rightarrow \infty$.

3. c. $a_N = N^N/N!$. Stirling's formula gives $N! \sim \sqrt{2\pi N} N^N e^{-N}$; therefore

$$a_N = \frac{N^N}{N!} \sim \frac{e^N}{\sqrt{2\pi N}}.$$

$$a_{10} \approx 2779; a_{100} \approx 10^{42}.$$

3. d. Stirling's formula gives

$$a_n \sim \frac{N^n e^n}{\sqrt{2\pi n} n^n} = \frac{1}{\sqrt{2\pi n}} \left(\frac{eN}{n} \right)^n.$$

Therefore, if $eN/n < 1$, then $a_n < 1$. We can rewrite $eN/n < 1$ as $eN < n$.

4. A sledgehammer works nicely here. Use Mathematica to evaluate $1000!/10^n$ for various n . If this expression ends in k zeros, then $1000!$ itself must end in $n+k$ zeros. For example, $1000!/10^{245}$ ends in 4 zeros, so $1000!$

ends in 249 zeros. The following table identifies the factors that contribute zeros, the number of such factors, and the total number of zeros contributed. (For example, 125 contributes 3 zeros when it is multiplied by 8 or 64.) The last three lines give the factors reduced modulo 100.

factor	zeros	total zeros
1000, 750, 500, 250, 125	3	15
100 – 400, 600 – 900	2	16
25*, 50, 75	2	58
10 – 40, 60 – 90	2	80
5, 15, 35, 45, 55, 65, 85, 95	2	80
Total		249

* excluding 125

The Binomial Distribution

5. Stirling's formula gives

$$\frac{(2n)!}{(n!)^2} \left(\frac{1}{2} \right)^{2n} \sim \frac{\sqrt{4\pi n} (2n)^{2n} e^{-2n}}{2\pi n \cdot n^{2n} e^{-2n} \cdot 2^{2n}} = \frac{1}{\sqrt{\pi n}}.$$

When $n = 50$, this approximation is $1/\sqrt{50\pi} = .0797885$. The exact value given by the binomial distribution is .0795892.

6. From the formula for P on page 774 of the text, with $p = k/n$,

$$\begin{aligned} P(n, k, p) &= \frac{n!}{k!(n-k)!} \left(\frac{k}{n} \right)^k \left(\frac{n-k}{n} \right)^{n-k} \\ &= \frac{n! k^k (n-k)^{n-k}}{k!(n-k)! n^n}. \end{aligned}$$

If we use Stirling's formula to replace $k!$, $(n-k)!$, and $n!$, we get

$$\begin{aligned} P(n, k, p) &\sim \frac{\sqrt{2\pi n} n^n e^{-n} k^k (n-k)^{n-k}}{\sqrt{2\pi k} k^k e^{-k} \sqrt{2\pi(n-k)} (n-k)^{n-k} e^{-(n-k)} n^n} \\ &= \frac{\sqrt{2\pi n}}{\sqrt{2\pi k} \sqrt{2\pi(n-k)}} = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{n}{k(n-k)}}. \end{aligned}$$

Now replace k by np :

$$\frac{n}{k(n-k)} = \frac{n}{np(n-np)} = \frac{1}{np(1-p)}.$$

Thus

$$P(n, k, p) \approx \frac{1}{\sqrt{2\pi np(1-p)}}.$$

To see if this is consistent with the answer to exercise 5, we must let $p = 1/2$ and replace n by $2n$. Then $p(1-p) = 1/4$ and

$$P(2n, n, 1/2) \approx \frac{1}{\sqrt{2\pi \cdot 2n/4}} = \frac{1}{\sqrt{\pi n}},$$

exactly as in exercise 5.

7. a. Let x_n be the position after n moves. Each move to the right adds $+1$ to x_n while each move to the left adds -1 to x_n . We are told that $x_n = r$, and that there have been k moves to the right (and $n-k$ moves to the left). Therefore

$$r = k \cdot (+1) + (n-k) \cdot (-1) = 2k - n.$$

Therefore $k = (n+r)/2$ and $n-k = (n-r)/2$. Since k is an integer, $n+r$ must be an *even* integer. Therefore n and r are either both even or both odd. In other words, after an even number of steps (n even), the particle must be an even distance r from the origin; after an odd number of steps, it must be an odd distance from the origin. Thus it can't be 3 or 7 units from the origin after 100 steps.

7. b. According to part a, the probability that the particle is exactly r units to the right after n steps is the same as the probability that there were $k = (n+r)/2$ steps to the right and $n-k = (n-r)/2$ steps to the left. Since the probability of going either left or right is $1/2$, this is formally the same as the coin-tossing problem. Thus the probability is

$$P = \frac{n!}{k!(n-k)!} \left(\frac{1}{2}\right)^n.$$

7. c. Stirling's formula converts the last expression into

$$\begin{aligned} P &\sim \frac{\sqrt{2\pi n}^{n+1/2} e^{-n}}{\sqrt{2\pi k}^{k+1/2} e^{-k} \sqrt{2\pi(n-k)}^{n-k+1/2} e^{-(n-k)} 2^n} \\ &= \frac{n^{n+1/2}}{\sqrt{2\pi} k^{k+1/2} (n-k)^{n-k+1/2} 2^n} \end{aligned}$$

$$\begin{aligned} \text{Now } k^{k+1/2} &= \frac{(n+r)^{(n+r+1)/2}}{2^{(n+r+1)/2}} \\ &= \frac{n^{(n+r+1)/2} (1+(r/n))^{(n+r+1)/2}}{2^{(n+r+1)/2}}. \end{aligned}$$

Similarly,

$$(n-k)^{n-k+1/2} = \frac{n^{(n-r+1)/2} (1-(r/n))^{(n-r+1)/2}}{2^{(n-r+1)/2}},$$

and the product of these two expressions is

$$\frac{n^{n+1} (1+(r/n))^{(n+r+1)/2} (1-(r/n))^{(n-r+1)/2}}{2^{n+1}}$$

If we substitute this into the expression for P , we get

$$\begin{aligned} &\frac{2}{\sqrt{2\pi n}^{1/2} (1+(r/n))^{(n+r+1)/2} (1-(r/n))^{(n-r+1)/2}} \\ &= \frac{\sqrt{2}}{\sqrt{\pi n} (1+(r/n))^{(n+r+1)/2} (1-(r/n))^{(n-r+1)/2}} \end{aligned}$$

7. d. The idea here is that $\ln a = b$ means $a = e^b$. Since we are assuming

$$\ln(1+x) \approx x - \frac{x^2}{2}, \quad \text{we have} \quad 1+x \approx e^{x-x^2/2}.$$

Now let $x = r/n$; then

$$(1+(r/n))^{(n+r+1)/2} \approx e^{((r/n)-r^2/(2n^2))(n+r+1)/2} = e^{Q_1}.$$

Now let $x = -r/n$; then

$$(1-(r/n))^{(n-r+1)/2} \approx e^{(-r/n-r^2/(2n^2))(n-r+1)/2} = e^{Q_2}.$$

The product is $e^{Q_1+Q_2}$, where

$$\begin{aligned} Q_1 + Q_2 &= \frac{n+1}{2} \left(\frac{r}{n} - \frac{r^2}{2n^2} - \frac{r}{n} - \frac{r^2}{2n^2} \right) \\ &\quad + \frac{r}{2} \left(\frac{r}{n} - \frac{r^2}{2n^2} + \frac{r}{n} + \frac{r^2}{2n^2} \right) \\ &= \frac{n+1}{2} \left(-\frac{r^2}{n^2} \right) + \frac{r^2}{n} \\ &= -\frac{r^2}{2n} - \frac{r^2}{2n^2} + \frac{r^2}{n} \\ &= \frac{r^2}{2n} \left(1 - \frac{1}{n} \right) \approx \frac{r^2}{2n} \end{aligned}$$

If we use the last expression to approximate the product in the denominator of P , we get

$$P \approx \sqrt{\frac{2}{\pi n}} e^{-r^2/2n}.$$

7. e. To see how the last result implies exercise 5, note that a head in the coin-toss corresponds to a step to the right, and a tail to a step to the left. If exactly half the

tosses are heads, then exactly half the steps are to the right—and the other half are to the left—so $r = 0$. We must also replace n by $2n$ (because exercise 5 deals with $2n$ coin flips). Thus

$$P \approx \sqrt{\frac{2}{\pi \cdot 2n}} e^{0/4n} = \frac{1}{\sqrt{\pi n}},$$

and this agrees with the result of exercise 5.

7. f. The only possibilities are that $r = -4, -2, 0, 2$, and 4 . The probability we want is the sum of these individual probabilities. Since the probabilities for r and $-r$ are the same, we get

$$P \approx \sqrt{\frac{2}{100\pi}} \left(2e^{-16/200} + 2e^{-4/200} + e^0 \right) = .383514$$

12.2 The Poisson Distribution

Probability Models

1. The possible outcomes of the first step are H (heads) and T (tails). If the first step yields H, then a die is rolled, and its outcomes are 1, 2, 3, 4, 5, and 6. This produces the combined outcomes H1, H2, H3, H4, H5, and H6. If the first step yields T, then the coin is flipped again, yielding H or T. Here the combined outcomes are HH and HT. Since the probability of a single H or T is $1/2$, and the probability of a 1, 2, 3, 4, 5, or 6 is $1/6$, the independence of the coin-tosses and the die rolls yields this model for the outcomes:

H1	H2	H3	H4	H5	H6	HH	HT
$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$

Note that the probability that *some* event happens is $6 \times (1/12) + 2 \times (1/4) = 1$, as it should be.

2. Since the coins are identical, the outcomes are 2 heads, 1 head, or 0 heads. However, to calculate the probabilities, we need to distinguish between the coins: the outcomes then are HH, HT, TH, TT, each with probability $1/4$. Since HT and TH both yield the observed outcome of 1 head, our model is this:

2 heads	1 head	0 heads
$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

3. a. Call the outcomes A and B, with probabilities .29 and $1 - .29 = .71$. The model is then just

A	B
.29	.71

3. b. Since the individual particles are indistinguishable, the outcomes can be described as 2 A, 1 A, and 0 A. But this is like the coin-toss experiment of exercise 2: when we distinguish between atoms, the outcomes are AA, AB, BA, BB. The probabilities are the products of the individual probabilities, so

AA	AB	BA	BB
$.29^2$	$.29 \times .71$	$.71 \times .29$	$.71^2$

The probability model for the disintegration of two particles is therefore

2 A	1 A	0 A
.0841	.4118	.5041

The Poisson Distribution

4. a. Since $p_n = \lambda^n e^{-\lambda} / n!$,

$$n \cdot p_n = n\lambda \frac{\lambda^{n-1} e^{-\lambda}}{n!} = \lambda \frac{\lambda^{n-1} e^{-\lambda}}{(n-1)!} = \lambda p_{n-1}.$$

4. b. Let $E_N = 0 p_0 + 1 p_1 + 2 p_2 + \cdots + N p_N$. Then

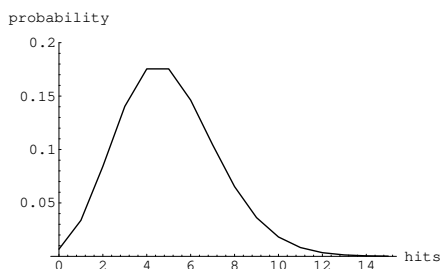
$$\begin{aligned} E &= \lim_{N \rightarrow \infty} E_N = \lim_{N \rightarrow \infty} 0 p_0 + 1 p_1 + 2 p_2 + \cdots + N p_N \\ &= \lim_{N \rightarrow \infty} \lambda p_0 + \lambda p_1 + \cdots + \lambda p_{N-1} \\ &= \lambda \lim_{N \rightarrow \infty} p_0 + p_1 + \cdots + p_{N-1} = \lambda \cdot 1 = \lambda \end{aligned}$$

5. The Poisson model is most appropriate when there are relatively few rain drops. Thus, use a Poisson model for a sprinkle, a linear model for a downpour.

6. Assuming that the single sector with 5 or more hits actually had 5 hits, the total number of hits is 535, and the number of hits per sector is $\lambda = 535/576 = .928819$. The following table records the actual frequencies and probabilities together with those predicted by the Poisson model with $\lambda = .928819$.

hits	actual		Poisson	
	freq	prob	freq	prob
0	229	.398	227.5	.395
1	211	.366	211.3	.367
2	93	.161	98.1	.170
3	35	.061	30.4	.053
4	7	.012	7.1	.012
5	1	.002	1.3	.002

7. A square kilometer is 10^6 square meters, so the average density of hits converts to 5 per square kilometer. Thus we take $\lambda = 5$ in a Poisson model. The following graph gives the theoretical probability for different numbers of hits in a square kilometer. the graph was drawn by the Mathematica 3 command given below it.



```
ListPlot[
  N[
    Table[
      {n, Exp[-5] 5^n/n!}, {n, 0, 15}
    ],
  ],
  PlotJoined -> True, PlotRange -> {0, .2},
  AxesLabel -> {"hits", "probability"}
]
```

8. Use a Poisson model where $\lambda = 1$ instruction per millisecond. We want to know the probability that more than 10 instructions happen in one millisecond. This is

$$\begin{aligned}
 & 1 - (p_0 + p_1 + \cdots + p_{10}) \\
 &= 1 - e^{-1} \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{10!} \right) \\
 &\approx 1 - 0.99999989952 = 1.0048 \times 10^{-8}
 \end{aligned}$$

—an extremely small number.

12.3 The Power Spectrum

The Problem of Phase

$$\begin{aligned}
 1. \quad \sin(bt - \varphi) &= \sin(bt) \cos(-\varphi) + \cos(bt) \sin(-\varphi) \\
 &= (\cos \varphi) \cdot \sin(bt) + (-\sin \varphi) \cdot \cos(bt),
 \end{aligned}$$

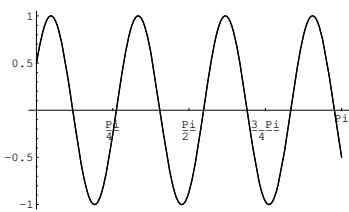
so $M = \cos \varphi$ and $N = -\sin \varphi$.

$$2. \text{ a. } \sin(5t - \pi/3) = \frac{1}{2} \sin(5t) - \frac{\sqrt{3}}{2} \cos(5t).$$

$$2. \text{ b. Take } A = 1, b = 7, \text{ and } \varphi = -\pi/6; \text{ then}$$

$$A \sin(bt - \varphi) = \sin(7t + \pi/6) = \frac{\sqrt{3}}{2} \sin(7t) + \frac{1}{2} \cos(7t).$$

The graphs look like this when plotted over $0 \leq t \leq \pi$:



2. c. Let $\varphi = -\arctan 2$. Then $\sin \varphi = -2/\sqrt{5}$ and $\cos \varphi = 1/\sqrt{5}$, so

$$\begin{aligned}
 f(t) &= \sqrt{5}(\cos \varphi \sin(t) - \sin \varphi \cos(t)) \\
 &= \sqrt{5} \sin(t - \varphi) = A \sin(bt - \varphi)
 \end{aligned}$$

when $A = \sqrt{5}$, $b = 1$ and $\varphi = -\arctan 2$.

3. This is just exercise 1 with the factor A added, so $M = A \cos \varphi$, $N = -A \sin \varphi$, and hence $M^2 + N^2 = A^2$.

4. Express both functions in the form $A \sin(t - \varphi)$:

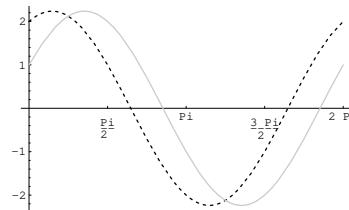
$$\sin(t) + 2 \cos(t) = \sqrt{5} \sin(t - \varphi_1), \quad \varphi_1 = -\arctan(2)$$

$$2 \sin(t) + \cos(t) = \sqrt{5} \sin(t - \varphi_2), \quad \varphi_2 = -\arctan(1/2)$$

The phase difference is $\varphi_1 - \varphi_2$. To determine this value, note that

$$\tan(\varphi_1 - \varphi_2) = \frac{\tan \varphi_1 - \tan \varphi_2}{1 + \tan \varphi_1 \tan \varphi_2} = \frac{2 - 1/2}{1 + 1} = \frac{3}{4},$$

so $\varphi_1 - \varphi_2 = \arctan(3/4) \approx .6435$ radians. In the figure below, the graph of $2 \sin(t) + \cos(t)$ is gray.



If, however, we plot $2 \sin(t + \varphi_1 - \varphi_2) + \cos(t + \varphi_1 - \varphi_2)$, then the gray graph shifts to the left and coincides with the dashed one. This demonstrates that the phase difference is indeed $\varphi_1 - \varphi_2$.

5. We want $A \sin(bx - \varphi) = -3 \sin(2x) - 4 \cos(2x)$. Thus

$$b = 2, A = \sqrt{M^2 + N^2} = 5, \varphi = \pi - \arctan(4/3).$$

We need to add π to φ in order to make $\cos \varphi$ negative.

6. The functions $\sin(x)$, $\sin(x+1)$, and $\sin(x+2)$ have the same period (viz. 2π) and amplitude, and differ only in phase. A little computer experimentation reveals that $\sin(x) + \sin(x+2)$ has the same phase as $\sin(x+1)$, so we can write

$$\sin(x) + \sin(x+2) = B \sin(x+1)$$

for some B , implying

$$\sin(x) + \sin(x+1) + \sin(x+2) = A \sin(x+1)$$

where $A = B + 1$. Finally, a further phase shift of π yields

$$\sin(x) + \sin(x+1) + \sin(x+2) + A \sin(x+1+\pi) = 0.$$

To complete the work we need to verify that $\sin(x) + \sin(x+2)$ does indeed have the same phase as $\sin(x+1)$, and then find B . Now $\sin(x+2) = \cos 2 \sin(x) + \sin 2 \cos(x)$, so

$$\begin{aligned} \sin(x) + \sin(x+2) &= (1 + \cos 2) \sin(x) + (\sin 2) \cos(x) \\ &= B \sin(x - \theta) \end{aligned}$$

for some B and θ . We know from the solutions to earlier exercises that $\tan \theta =$

$$\frac{-\sin 2}{1 + \cos 2} = \frac{-2 \sin 1 \cos 1}{1 + 2 \cos^2 1 - 1} = \frac{-\sin 1}{\cos 1} = -\tan 1.$$

This confirms that $\theta = -1$. (We used the “double-angle” formulas that say $\sin 2 = 2 \sin 1 \cos 1$ and $\cos 2 = 2 \cos^2 1 - 1$.) Also,

$$\begin{aligned} B &= \sqrt{(1 + \cos 2)^2 + \sin^2 2} \\ &= \sqrt{2 + 2 \cos 2} = \sqrt{2 + 4 \cos^2 1 - 2} = 2 \cos 1, \end{aligned}$$

so $A = 1 + 2 \cos 1$ and $\varphi = \theta \pm \pi = -1 \pm \pi$.

The Programs DETECTOR and POWER

7. Just change the line containing `deltaS`. To make `SDETECTOR`, add a factor of 2:

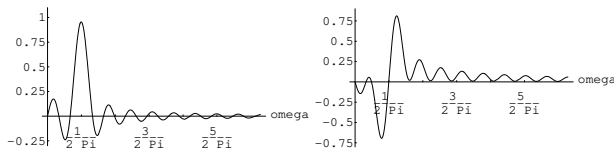
```
deltaS = 2 * (fnf(t) * SIN(twopi * ...etc.
```

To make `CDETECTOR`, add a factor of 2 and change the sine to the cosine:

```
deltaS = 2 * (fnf(t) * COS(twopi * ...etc.
```

8. a. `POWER` does not distinguish between $f(t) = \sin(t)$ and $g(t) = \cos(t)$, and we shouldn't expect it to, because these two functions differ only in phase ($\cos(t) = \sin(t + \pi/2)$). In both cases `POWER` produces a single peak of height ≈ 1 at $\omega = 1/2\pi$.

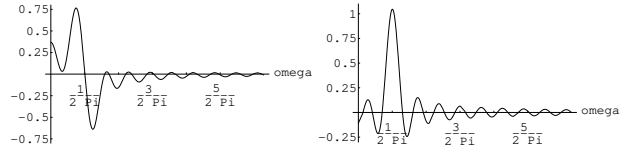
8. b. `SDETECTOR` does distinguish between f and g . The graph for f (on the left, below) has a strong peak at $\omega = 1/2\pi$, while the graph for g is zero at $\omega = 1/2\pi$.



Note: we get explicit formulas in this case. For example, $g(t)$ on the interval $0 \leq t \leq 10$ yields $D_s(\omega) =$

$$\frac{2\pi\omega - \sin(10) \sin(20\pi\omega) - 2\pi\omega \cos(10) \cos(20\pi\omega)}{5(4\pi^2\omega^2 - 1)}.$$

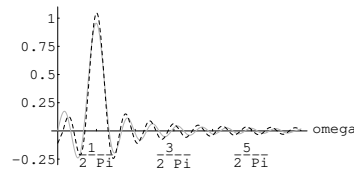
8. c. `CDETECTOR` also distinguishes between f and g . However, this time the graph for g (on the right, below) has the strong peak at $\omega = 1/2\pi$, while the graph for f is zero at $\omega = 1/2\pi$.



The explicit formula for $D_c(\omega)$ for $f(t)$ on the interval $0 \leq t \leq 10$ is $D_c(\omega) =$

$$\frac{2\pi\omega \sin(10) \sin(20\pi\omega) + \cos(10) \cos(20\pi\omega) - 1}{5(4\pi^2\omega^2 - 1)}.$$

The outputs of f on D_s and g on D_c are quite similar, but they are not identical. The two output graphs, when superimposed, look like this (D_s is gray):

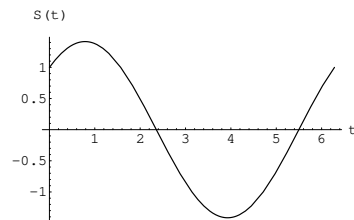


The formulas for these two functions are

$$D_s(\omega) = \frac{\cos(10) \sin(20\pi\omega) - 2\pi\omega \sin(10) \cos(20\pi\omega)}{5(4\pi^2\omega^2 - 1)}$$

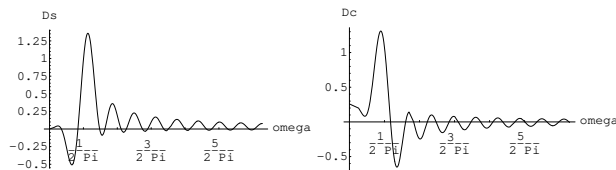
$$D_c(\omega) = \frac{2\pi\omega \cos(10) \sin(20\pi\omega) - \sin(10) \cos(20\pi\omega)}{5(4\pi^2\omega^2 - 1)}.$$

9. a. The power spectrum of S has a single peak of height ≈ 1.45 at $\omega = 1/2\pi$. This is what we would expect, because $S(t) = \cos(t) + \sin(t)$ has a frequency of $1/2\pi$ and an amplitude of ≈ 1.4 .



9. b. The spectrum of S is essentially the spectrum of f (or g) stretched vertically by a factor of about 1.45.

9. c. Both graphs show strong signals near $\omega = 1/2\pi$. $D_s(\omega)$ reaches a peak of about 1.4 at $\omega \approx .18$ (below left), while $D_c(\omega)$ has a peak of about 1.3 at $\omega \approx .14$.

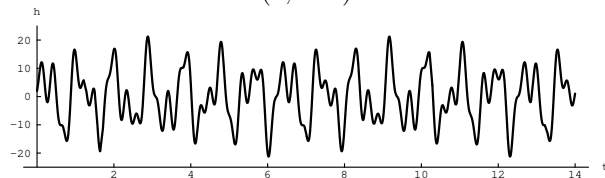


These outputs appear to be a mixture of the outputs of f and g . Indeed, that is exactly the case. To show this, write $D_s(\omega; S)$ to denote the sine detector for S ; then

$$\begin{aligned} D_s(\omega; S) &= \frac{1}{5} \int_0^{10} (\sin(t) + \cos(t)) \sin(2\pi\omega t) dt \\ &= \frac{1}{5} \int_0^{10} \sin(t) \sin(2\pi\omega t) dt + \frac{1}{5} \int_0^{10} \cos(t) \sin(2\pi\omega t) dt \\ &= D_s(\omega; f) + D_s(\omega; g). \end{aligned}$$

Similarly, $D_c(\omega; S) = D_c(\omega; f) + D_c(\omega; g)$.

10. a. The student's graph may have different proportions from the one in the text. This is true in the following example, where the t -axis has also been shifted down so it intersects the h -axis at $(0, -25)$.



10. b. When POWER is run with

```
omegasteps = 2 ^ 10
numberofsteps = 2 ^ 7
```

there are three peaks at $\omega \approx 7/2\pi$ (height 10), $\omega \approx 13/2\pi$ (height 7), and $\omega \approx 23/2\pi$ (height 5). This agrees with the text.

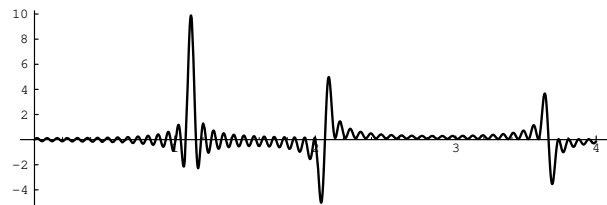
However, if POWER is run with the parameter values given in the program—namely

```
omegasteps = 2 ^ 9
numberofsteps = 2 ^ 6
```

then three additional, spurious, peaks appear. The spurious peaks occur at $\omega \approx .9$ (height 5.5), $\omega \approx 2.5$ (height 7), and $\omega \approx 3.5$ (height 10). This illustrates the need to recalculate until the output stabilizes.

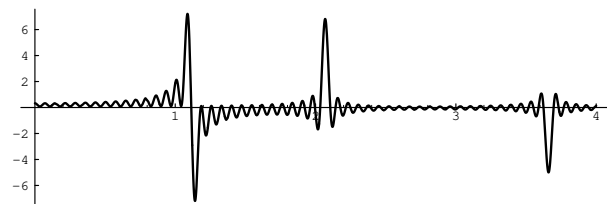
11. We need to distinguish between the bump caused by a signal that is *in phase* with the detector, and one that is

out of phase. The “in-phase” bump is the one on the top of page 807 of the text (which shows the cosine detector detecting a cosine function). The “out-of-phase” bump is the one above it; it shows the sine detector detecting the same cosine function.



The graph of D_s , above, has an “in-phase” bump at $\omega = 7/2\pi$ with height 10, indicating the presence of a term of the form $10 \sin(7t)$ in $h(t)$. It has “out-of-phase” bumps at $\omega = 13/2\pi$ and $\omega = 23/2\pi$, indicating the presence of terms that are multiples of $\cos(13t)$ and $\cos(23t)$. The heights are unreliable indicators of the multipliers, though. The next exercise will settle this matter.

12. The graph of D_c , below, has an “out-of-phase” bump at $\omega = 7/2\pi$, indicating the presence of a term of the form $M \sin(7t)$ in $h(t)$. (The height of the bump is a poor indicator of the value of M .) It has an “in-phase” bump at $\omega = 13/2\pi$ (height 7), indicating the presence of a term of the form $7 \cos(13t)$. It has another “in-phase” bump at $\omega = 23/2\pi$ (height 5), but this bump is “upside-down”. This implies that $h(t)$ has a term of the form $-5 \cos(23t)$. (Note that, in the graph of D_s in exercise 11, above, the “out-of-phase” bump at $\omega = 23/2\pi$ is *also* upside-down—providing further confirmation that $\cos(23t)$ appears with a negative coefficient.)



A Grain of Salt

13. The power spectrum has a single peak of height 1 at $\omega \approx 1$.

14. The power spectrum has a single peak of height 1 at $\omega \approx 1$. The peak here is sharper than the previous graph.

15. A new peak of height 1 appears at $\omega \approx 7/3$.

16. There are four peaks in all: two new peaks of height 1 appear at $\omega \approx 2/3$ and $\omega \approx 8/3$.

17. CDETECTOR has a value of 0 when $\omega = 7/3$.

18. We use the formula $\int_a^b \sin \alpha x \sin \beta x dx =$

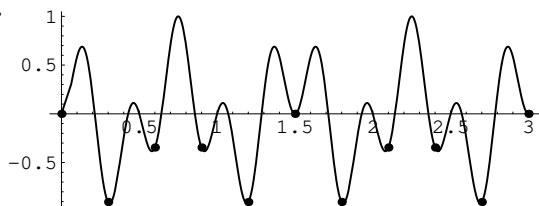
$$\frac{1}{\beta^2 - \alpha^2} (\alpha \cos \alpha x \sin \beta x - \beta \sin \alpha x \cos \beta x) \Big|_a^b$$

Thus $\int_0^{30} \sin(2\pi x) \sin(14\pi x/3) dx =$

$$\frac{9}{160\pi^2} \left(2\pi \cos(60\pi) \sin(140\pi) - \frac{14\pi}{3} \sin(60\pi) \cos(140\pi) \right)$$

Since $\sin(60\pi) = \sin(140\pi) = 0$, the entire expression equals 0, so the exact value of the integral is zero.

19.



It appears that the graph $h(x) = \sin(2\pi x) \sin(2\pi \frac{7}{3}x)$ completes two cycles on $[0, 3]$, implying the period is 1.5. We can *prove* this is true by a direct check:

$$\begin{aligned} h(x + \tfrac{3}{2}) &= \sin(2\pi x + 3\pi) \sin(\tfrac{14}{3}\pi x + 7\pi) \\ &= -\sin(2\pi x) \cdot -\sin(\tfrac{14}{3}\pi x) = h(x). \end{aligned}$$

20.

x	$h(x)$
0	0
.3	-.904508 = $-\sin^2(2\pi/5)$
.6	-.345492 = $-\sin^2(\pi/5)$
.9	-.345492 = $-\sin^2(\pi/5)$
1.2	-.904508 = $-\sin^2(2\pi/5)$
1.5	0

These values are all negative (or zero); they are plotted on the previous graph. We also tabulate the exact values, which we obtain by exploiting trigonometric identities for the sine function. For example,

$$\begin{aligned} h(.6) &= \sin(2\pi \tfrac{6}{10}) \sin(\tfrac{14}{3}\pi \tfrac{6}{10}) = \sin(\tfrac{6}{5}\pi) \sin(\tfrac{14}{5}\pi) \\ &= \sin(\pi + \tfrac{1}{5}\pi) \sin(2\pi + \tfrac{4}{5}\pi) \\ &= -\sin(\tfrac{1}{5}\pi) \sin(\tfrac{4}{5}\pi) = -\sin^2(\tfrac{1}{5}\pi) \end{aligned}$$

The Riemann sum we are asked to construct is

$$\begin{aligned} S &= .3[h(0) + h(.3) + h(.6) + h(.9) + h(1.2)] \\ &= .3[2h(.3) + 2h(.6)] \\ &= -.3[1.809016 + .690984] = -.750000 \end{aligned}$$

In fact, the value of the Riemann sum is *exactly* $-3/4$. We can show this if we can show

$$-h(.3) - h(.6) = \sin^2(\tfrac{\pi}{5}) + \sin^2(\tfrac{2\pi}{5}) = \tfrac{5}{4},$$

for then $S = -\tfrac{3}{5}[-h(.3) - h(.6)] = -3/4$. The first step is

$$\begin{aligned} \sin(\tfrac{\pi}{5}) &= \sin(\tfrac{4\pi}{5}) = 2\sin(\tfrac{2\pi}{5})\cos(\tfrac{2\pi}{5}) \\ &= 4\sin(\tfrac{\pi}{5})\cos(\tfrac{\pi}{5})\cos(\tfrac{2\pi}{5}). \end{aligned}$$

Now divide by $4\sin(\tfrac{\pi}{5})$ to get

$$\tfrac{1}{4} = \cos(\tfrac{\pi}{5})\cos(\tfrac{2\pi}{5}).$$

The second step uses the trigonometric identity $2\cos(A)\cos(B) = \cos(A-B) + \cos(A+B)$:

$$\begin{aligned} 2\cos(\tfrac{\pi}{5})\cos(\tfrac{2\pi}{5}) &= \cos(\tfrac{\pi}{5}) + \cos(\tfrac{3\pi}{5}) \\ &= \cos(\tfrac{\pi}{5}) - \cos(\tfrac{2\pi}{5}) \end{aligned}$$

Thus $4\cos^2(\tfrac{\pi}{5})\cos^2(\tfrac{2\pi}{5}) =$

$$\cos^2(\tfrac{\pi}{5}) - 2\cos(\tfrac{\pi}{5})\cos(\tfrac{2\pi}{5}) + \cos^2(\tfrac{2\pi}{5})$$

Since $\cos(\tfrac{\pi}{5})\cos(\tfrac{2\pi}{5}) = 1/4$, the last equation reduces to

$$\tfrac{1}{4} = \cos^2(\tfrac{\pi}{5}) - \tfrac{1}{2} + \cos^2(\tfrac{2\pi}{5}),$$

or

$$\cos^2(\tfrac{\pi}{5}) + \cos^2(\tfrac{2\pi}{5}) = \tfrac{3}{4}.$$

Finally, then,

$$\begin{aligned} \sin^2(\tfrac{\pi}{5}) + \sin^2(\tfrac{2\pi}{5}) &= 2 - (\cos^2(\tfrac{\pi}{5}) + \cos^2(\tfrac{2\pi}{5})) \\ &= 2 - \tfrac{3}{4} = \tfrac{5}{4}, \end{aligned}$$

as we claimed at the outset. (*This is just one of many different paths through the derivation.*)

21. We are given a step size of $\Delta x = .3$. Since $h(x)$ is periodic with period $T = 1.5$, an integral multiple of these steps (namely, five) will cover one period. Also, the interval $[0, 30]$ contains exactly 20 periods of h . Therefore, a left endpoint Riemann sum for

$$\int_0^{30} h(x) dx$$

with $\Delta x = .3$ will consist of 20 copies of the left endpoint Riemann sum for

$$\int_0^{1.5} h(x) dx$$

with the same Δx . The latter was found in the previous exercise to have the value $-3/4$. The first Riemann sum will therefore be 20 times $-3/4$. The left endpoint Riemann sum for

$$\frac{1}{15} \int_0^{30} h(x) dx$$

will be 20/15th of $-3/4$, or -1 .

22. The value of this integral is *exactly* zero, and this was shown using techniques from chapter 11. However, numerical integration with a left endpoint Riemann sum with 100 steps gives a value of -1 . The latter result shows up as a spurious spike when the power spectrum is calculated numerically with too few steps.

12.4 Fourier Series

1. a. When $m \neq k$ we can use the formula

$$\begin{aligned} & \int_{-\pi}^{\pi} \sin(kx) \cos(mx) dx \\ &= \frac{m \sin(kx) \sin(mx) + k \cos(kx) \cos(mx)}{m^2 - k^2} \Big|_{-\pi}^{\pi} \\ &= k \left(\frac{\cos(k\pi) \cos(m\pi) - \cos(-k\pi) \cos(-m\pi)}{m^2 - k^2} \right) \\ &= k \left(\frac{\cos(k\pi) \cos(m\pi) - \cos(k\pi) \cos(m\pi)}{m^2 - k^2} \right) \\ &= 0. \end{aligned}$$

When $m = k$ the integrand is $\sin(kx) \cos(kx) = \sin(2kx)/2$, so the value of the integral is

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{\sin(2kx)}{2} dx &= \frac{-\cos(2kx)}{4k} \Big|_{-\pi}^{\pi} \\ &= \frac{-\cos(2k\pi) + \cos(-2k\pi)}{4k} = 0. \end{aligned}$$

1. b. If $k = m$, the integrand is $\sin^2(kx)$ and the integral is

$$\int_{-\pi}^{\pi} \sin^2(kx) dx = \frac{x}{2} - \frac{\sin(2kx)}{4k} \Big|_{-\pi}^{\pi} = \frac{\pi}{2} - \frac{-\pi}{2} = \pi.$$

If $k \neq m$, then

$$\begin{aligned} & \int_{-\pi}^{\pi} \sin(kx) \sin(mx) dx \\ &= \frac{k \cos(kx) \sin(mx) - m \sin(kx) \cos(mx)}{m^2 - k^2} \Big|_{-\pi}^{\pi} \\ &= 0 \end{aligned}$$

because every term has a factor $\pm \sin(k\pi)$ or $\pm \sin(m\pi)$, and each of these is 0.

1. c. If $k = m = 0$, the integrand is 1 so the value of the integral is $1 \cdot 2\pi = 2\pi$.

If $k = m \neq 0$, the integrand is $\cos^2(kx) = 1 - \sin^2(kx)$, so the value of the integral is

$$\int_{-\pi}^{\pi} 1 dx - \int_{-\pi}^{\pi} \sin^2(kx) dx = 2\pi - \pi = \pi.$$

(Of course this can be computed directly, too.)

If $k \neq m$, then

$$\begin{aligned} & \int_{-\pi}^{\pi} \cos(kx) \cos(mx) dx = \\ & \frac{m \cos(kx) \sin(mx) - k \sin(kx) \cos(mx)}{m^2 - k^2} \Big|_{-\pi}^{\pi} = 0 \end{aligned}$$

because every term has a factor $\pm \sin(k\pi)$ or $\pm \sin(m\pi)$, and each of these is 0.

2. The triangular function is defined by the formula

$$f(x) = \begin{cases} \pi + x & x \leq 0 \\ \pi - x & x \geq 0 \end{cases}$$

The coefficients of the sine terms in the Fourier series are determined by the integral

$$\begin{aligned} & \int_{-\pi}^{\pi} f(x) \sin(kx) dx = \\ & \int_{-\pi}^0 (\pi + x) \sin(kx) dx + \int_0^{\pi} (\pi - x) \sin(kx) dx. \end{aligned}$$

We can use integration by parts. To do the first integral, take $F(x) = \pi + x$, $g(x) = \sin(kx)$. Then

$$\begin{aligned} & \int_{-\pi}^0 (\pi + x) \sin(kx) dx \\ &= \frac{-(\pi + x) \cos(kx)}{k} \Big|_{-\pi}^0 + \int_{-\pi}^0 \frac{\cos(kx)}{k} dx \\ &= \frac{-(\pi + x) \cos(kx)}{k} + \frac{\sin(kx)}{k^2} \Big|_{-\pi}^0 \\ &= -\frac{\pi}{k} - \left(\frac{-(\pi - \pi) \cos(k\pi)}{k} \right) = -\frac{\pi}{k}. \end{aligned}$$

Similarly, the second integral is

$$\begin{aligned} & \int_0^\pi (\pi - x) \sin(kx) dx \\ &= \left. \frac{-(\pi - x) \cos(kx)}{k} \right|_0^\pi - \int_0^\pi \frac{\cos(kx)}{k} dx \\ &= \frac{-(\pi - x) \cos(kx)}{k} + \frac{\sin(kx)}{k^2} \Big|_0^\pi \\ &= \frac{-(\pi - \pi) \cos(k\pi)}{k} - \frac{-\pi}{k} = +\frac{\pi}{k}. \end{aligned}$$

The sum of the values of the two integrals is 0, as required.

3. a. The Fourier series is

$$x = 2 \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \dots \right).$$

That is, $a_k = 0$, $b_k = (-1)^{k+1} \frac{2}{k}$. Use integration by parts to determine these:

$$\begin{aligned} \pi a_k &= \int_{-\pi}^\pi x \cos(kx) dx \\ &= \left. \frac{x \sin(kx)}{k} \right|_{-\pi}^\pi - \frac{1}{k} \int_{-\pi}^\pi \sin(kx) dx \\ &= \left. \frac{\cos(kx)}{k^2} \right|_{-\pi}^\pi = \frac{\cos(k\pi) - \cos(-k\pi)}{k^2} = 0. \end{aligned}$$

$$\begin{aligned} \pi b_k &= \int_{-\pi}^\pi x \sin(kx) dx \\ &= \left. \frac{-x \cos(kx)}{k} \right|_{-\pi}^\pi + \frac{1}{k} \int_{-\pi}^\pi \cos(kx) dx \\ &= \left. \frac{-x \cos(kx)}{k} + \frac{\sin(kx)}{k^2} \right|_{-\pi}^\pi \\ &= \frac{-(\pi \cos(k\pi)) + (-\pi \cos(-k\pi))}{k^2} \\ &= \frac{-2\pi \cos(k\pi)}{k^2} = -\frac{2\pi}{k} (-1)^k = (-1)^{k+1} \frac{2\pi}{k} \end{aligned}$$

3. b. the Fourier series is

$$\pi^2 - x^2 = \frac{2\pi^2}{3} + 4 \left(\cos x - \frac{\cos 2x}{4} + \frac{\cos 3x}{9} - \dots \right);$$

that is, $a_0 = \frac{2\pi^2}{3}$, $a_k = (-1)^{k+1} \frac{4}{k^2}$, $b_k = 0$. In fact,

$$2\pi a_0 = \int_{-\pi}^\pi \pi^2 - x^2 dx = \pi^2 x - \frac{x^3}{3} \Big|_{-\pi}^\pi = \frac{4\pi^3}{3}.$$

Integration by parts with $F(x) = \pi^2 - x^2$, $g(x) = \cos(kx)$ gives

$$\begin{aligned} \pi a_k &= \int_{-\pi}^\pi (\pi^2 - x^2) \cos(kx) dx \\ &= \left. \frac{(\pi^2 - x^2) \sin(kx)}{k} \right|_{-\pi}^\pi + \frac{2}{k} \int_{-\pi}^\pi x \sin(kx) dx \\ &= 0 + (-1)^{k+1} \frac{2}{k} \cdot \frac{2\pi}{k} = (-1)^{k+1} \frac{4\pi}{k^2} \end{aligned}$$

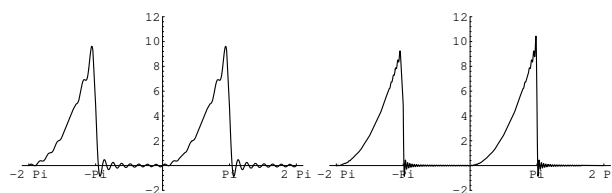
Here we took the value of $\int x \sin(kx) dx$ from the solution to part a.

$$\begin{aligned} \pi b_k &= \int_{-\pi}^\pi (\pi^2 - x^2) \sin(kx) dx \\ &= \left. \frac{-(\pi^2 - x^2) \cos(kx)}{k} \right|_{-\pi}^\pi - \frac{2}{k} \int_{-\pi}^\pi x \cos(kx) dx \\ &= 0 - \frac{2}{k} \left(\frac{x \sin(kx)}{k} + \frac{\cos(kx)}{k^2} \right) \Big|_{-\pi}^\pi \\ &= \frac{2}{k^3} (\cos(k\pi) - \cos(-k\pi)) = 0. \end{aligned}$$

3. c. The Fourier series is rather bizarre; the coefficients are

$$\begin{aligned} a_0 &= \frac{\pi^2}{6} & a_k &= (-1)^k \frac{2}{k} \\ b_k &= (-1)^{k+1} \frac{\pi}{k} + ((-1)^k - 1) \frac{2}{\pi k^3}. \end{aligned}$$

Sketched below are the approximations up to $k = 15$ (on the left) and $k = 60$. They do indeed appear to look like $y = x^2$ on one side and $y = 0$ on the other. Of course, there is a discontinuity in the original function at $x = \pm\pi$, so the Fourier approximations are jagged there (the Gibbs phenomenon).



As usual, we determine the coefficients using integration by parts. Because of the way $f(x)$ is defined, we can

restrict all integrals to $[0, \pi]$.

$$\begin{aligned}
 \pi a_k &= \int_0^\pi x^2 \cos(kx) dx \\
 &= \frac{x^2 \sin(kx)}{k} \Big|_0^\pi - \frac{2}{k} \int_0^\pi x \sin(kx) dx \\
 &= 0 - \frac{2}{k} \left(\frac{-x \cos(kx)}{k} \Big|_0^\pi + \frac{1}{k} \int_0^\pi \cos(kx) dx \right) \\
 &= -\frac{2}{k} \left(\frac{-\pi \cos(k\pi)}{k} - \frac{\sin(kx)}{k^2} \Big|_0^\pi \right) = (-1)^k \frac{2\pi}{k^2} \\
 \\
 \pi b_k &= \int_0^\pi x^2 \sin(kx) dx \\
 &= \frac{-x^2 \cos(kx)}{k} \Big|_0^\pi + \frac{2}{k} \int_0^\pi x \cos(kx) dx \\
 &= \frac{-\pi^2 \cos(k\pi)}{k} + \frac{2}{k} \left(\frac{x \sin(kx)}{k} \Big|_0^\pi - \frac{1}{k} \int_0^\pi \sin(kx) dx \right) \\
 &= (-1)^{k+1} \frac{\pi^2}{k} + \frac{2}{k} \frac{\cos(kx)}{k^2} \Big|_0^\pi \\
 &= (-1)^{k+1} \frac{\pi^2}{k} + \frac{2}{k^3} (\cos(k\pi) - 1) \\
 &= (-1)^{k+1} \frac{\pi^2}{k} + ((-1)^k - 1) \frac{2}{k^3}
 \end{aligned}$$

4. In May's model we take $T = 38.6$ days and assume the initial conditions $x = 7.75$ and $y = 2.38$. If $y(t)$ denotes the solution of May's model for the predator, then Fourier polynomials that approximate $y(t)$ have the form

$$\phi(t) = a_0 + \sum_{k=1}^n \left(a_k \cos \frac{2k\pi t}{T} + b_k \sin \frac{2k\pi t}{T} \right).$$

where the Fourier coefficients are given by

$$\begin{aligned}
 a_0 &= \frac{1}{38.6} \int_0^{38.6} y(t) dt \\
 a_k &= \frac{2}{38.6} \int_0^{38.6} y(t) \cos \left(\frac{2k\pi t}{T} \right) dt \\
 b_k &= \frac{2}{38.6} \int_0^{38.6} y(t) \sin \left(\frac{2k\pi t}{T} \right) dt
 \end{aligned}$$

We need to solve the differential equations in May's model and then calculate the integrals. Both of these tasks are to be done numerically; the following QuickBasic program does all this, using Simpson's rule to calculate the integrals. Notice there are two places where the command continues to a second line.

```
DEFDBL A-Z
DIM x(2000), y(2000)
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x(0) = 7.75
y(0) = 2.38
c = 0
d = 38.6
period = 38.6
pi = 4 * ATN(1)
f = 2 * pi / period
numberofsteps = 1000
deltat = (d - c) / numberofsteps
alc = 0
amc = 0
arc = 0
als = 0
ams = 0
ars = 0
FOR k = 0 TO 2 * numberofsteps - 1
    xprime = .6 * x(k) * (1 - x(k) / 10)
        - .5 * x(k) * y(k) / (x(k) + 1)
    yprime =
        .1 * y(k) * (1 - y(k) / (2 * x(k)))
    deltax = xprime * deltat / 2
    deltay = yprime * deltat / 2
    x(k + 1) = x(k) + deltax
    y(k + 1) = y(k) + deltay
NEXT k
FOR m = 0 TO 2 * numberofsteps - 2 STEP 2
    deltaAl = y(m) * deltat
    t = t + deltat / 2
    deltaAm = y(m + 1) * deltat
    t = t + deltat / 2
    deltaAr = y(m + 2) * deltat
    alc = alc + deltaAl
    amc = amc + deltaAm
    arc = arc + deltaAr
NEXT m
simpson = (alc + 4 * amc + arc) / 6
PRINT (1 / period) * simpson
FOR j = 0 TO 6
    t = 0
    alc = 0
    als = 0
    amc = 0
    ams = 0
    arc = 0
    ars = 0
    FOR l = 0 TO 2 * numberofsteps - 2 STEP 2
        deltaAlc =
            COS(j * f * t) * y(l) * deltat
        deltaAls =
            SIN(j * f * t) * y(l) * deltat
        t = t + deltat / 2
        deltaAmc =
```

```

      COS(j * f * t) * y(l + 1) * deltat
deltaAms =
      SIN(j * f * t) * y(l + 1) * deltat
t = t + deltat / 2
deltaArc =
      COS(j * f * t) * y(l + 2) * deltat
deltaArs =
      SIN(j * f * t) * y(l + 2) * deltat
alc = alc + deltaAlc
als = als + deltaAls
amc = amc + deltaAmc
ams = ams + deltaAms
arc = arc + deltaArc
ars = ars + deltaArs
NEXT l
simpsonc = (alc + 4 * amc + arc) / 6
simpsons = (als + 4 * ams + ars) / 6
PRINT j, (2 / period) * simpsonc,
        (2 / period) * simpsons
NEXT j

```

This program produces the following output:

k	a_k	b_k
0	2.6053	
1	.0607	2.0469
2	-.4913	-.5950
3	.3441	.1510
4	-.2085	-.0026
5	.1147	-.0476
6	-.0546	.0530

The graph shows in gray is the Fourier polynomial that has these coefficients; the direct numerical solution to the initial value problem is shown dashed.

