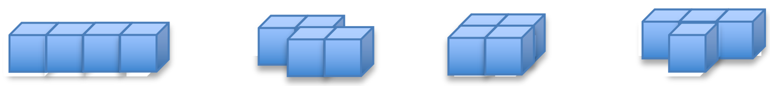
Optimizing Area, Perimeter, and Surface Area **Name** ___Solutions_____

One of the most common applications of mathematics is to optimize a situation, which means to find maximum or minimum possible solution. For example, if you are designing a car, you might want it to be most fuel efficient, which means to use the minimum amount of energy possible to go a specific distance. Or you might want it to maximize how fast it can travel or to minimize the cost to produce it.

For each of the following problems, be thorough. Consider numerous examples for each problem. Count all surfaces in all directions.

1. Use 4 cubic centimeters (unit cm^3). For each question, all of the cubes have to be used, have to be part of one continuous object, and each cube has to share at least one full square face with at least one other cube.
    a. Put the cubes together in one shape and determine the surface area (including those resting on the table).



    b. What shape will have the **greatest possible surface** area for your set of 4 cubes? Describe it and report both the volume and surface area.

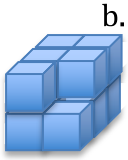
    All arrangements other than the square have a V of 4 cm^3 and a SA of 18 cm^2 (each has 6 internal/covered faces of the original 24 from the 4 cubes).

    c. What shape will have the **least possible surface** area for your set of 4 cubes? Describe it and report both the volume and surface area.

    The square pattern has a V of 4 cm^3 and a SA of 16 cm^2 (with 8 internal faces).

2. You have been given a handful of cubic centimeters (unit cm^3). For each question, all of the cubes have to be used, have to be part of one continuous object, and have to share at least one full square face with at least one other cube.
    a. What shape will have the **greatest possible surface** area for your handful of cubes? Describe it and report both the volume and surface area.

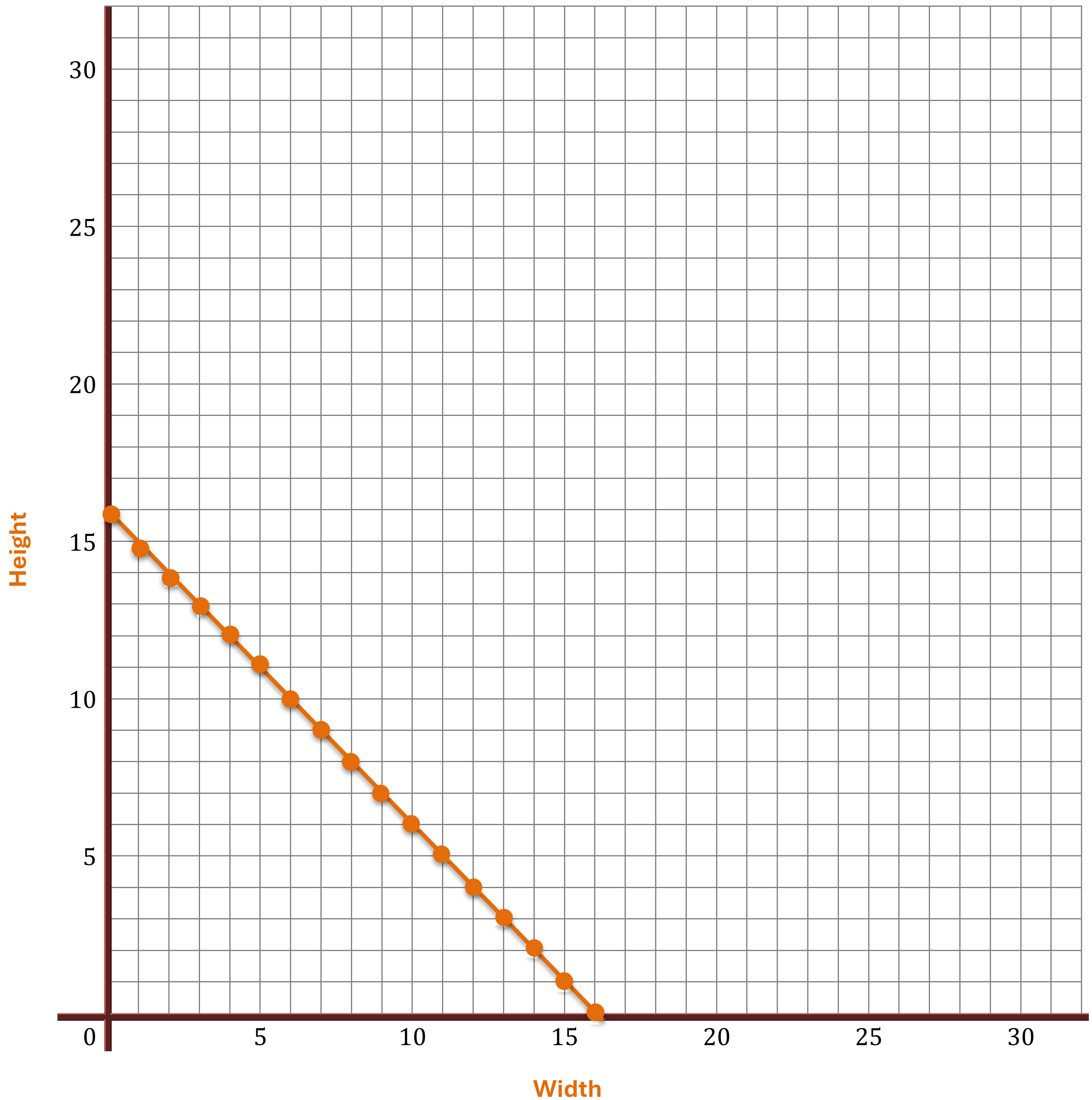
    I was given 11 cubes with a V of 11 cm^3. Lining them all up gives a SA of 46 cm^2. Any arrangement that doesn't wrap back around would be the same (you have to have no more than two neighbors and two end cubes should only have one).



    b. What shape will have the **least possible surface** area for your handful of cubes? Describe it and report both the volume and surface area.
    I think making as small a box as possible is best, so I made a 2x2x3 box with one corner removed giving a V of 11 cm^3 and a SA of 32 cm^2. This shape has the same SA as the 12 cubed 2x2x3 box (the three newly exposed inside faces replacing the original ones).

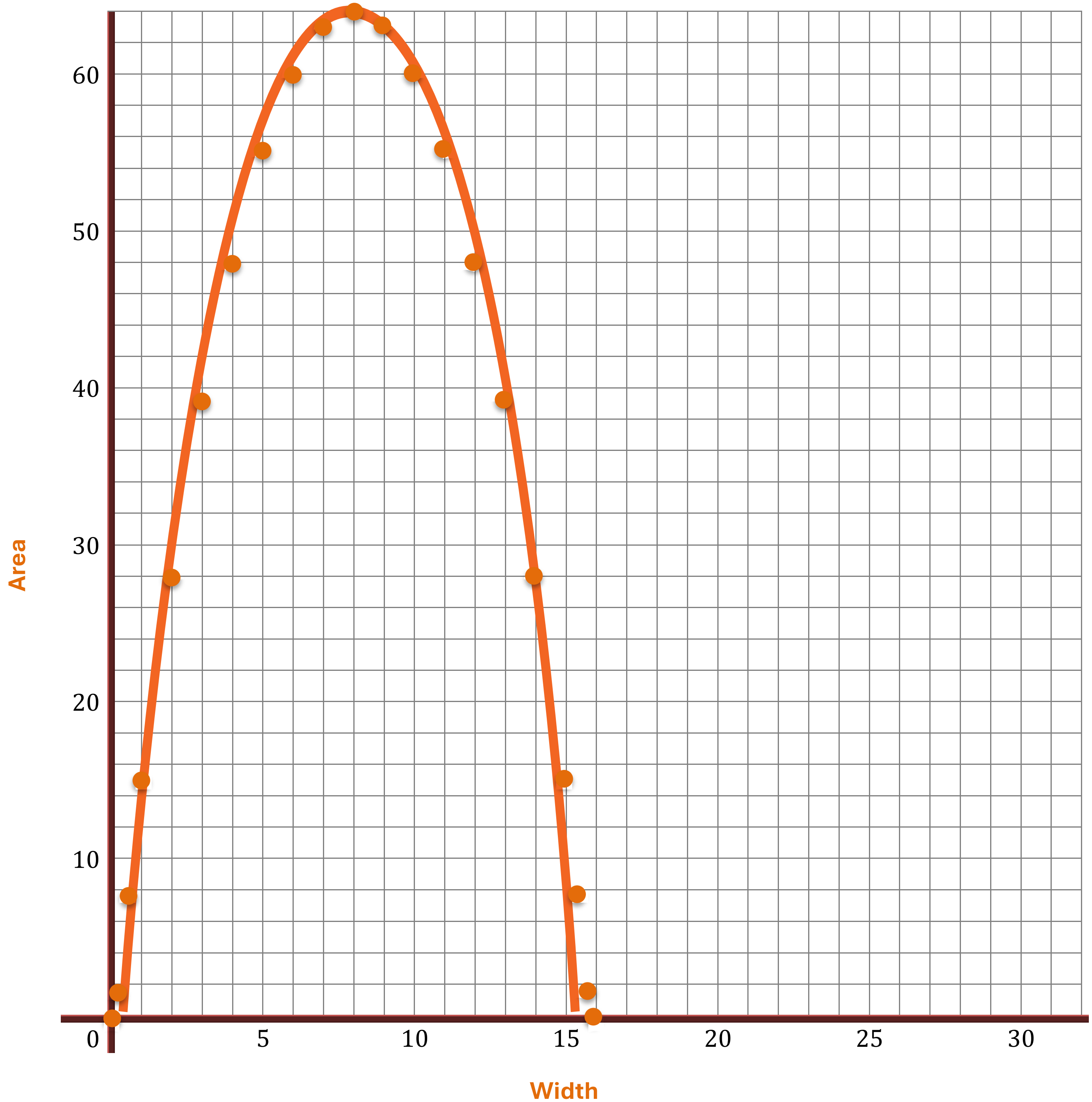
3. You have a **rectangle** with a **perimeter of 32 cm**.
    a. Fill the table below with many possible dimensions for your rectangle. **Don't limit yourself to whole numbers.** You can use the centimeter grid at right to draw some of your rectangles.
    b. Graph the values on the graph on the next page. **Label** the x-axis "width" and the y-axis "height".

Width	**Height**	**Perimeter**	**Area**
8	8	32	64
9	7	32	63
10	6	32	60
11	5	32	55
12	4	32	48
13	3	32	39
14	2	32	28
15	1	32	15
16	0	32	0
0.5	15.5	32	7.75
0.1	15.9	32	1.59
I also graphed the reverse of all of the above.		32	
		32	
		32	



I connected the points, because all values in between also make a rectangle with a perimeter of 32.

4. Return to the first page and complete the area column for each rectangle.
5. Plot points on the graph below with your *x*-axis being the **width** of each rectangle and the *y*-axis being that rectangle's **area**. Label both axes. Note that the *y*-axis has a different scale with each box 2 units high (but still only one unit wide).



6. Studying the graph for problem 5., determine the dimensions of the rectangle that had the greatest area. If you are not sure that rectangle's point is graphed, use the points you have to explore the region where the maximum seems to be and find it.

   The greatest area is when we have an 8x8 square.

7. Why is that rectangle the one with the most area for the fixed amount of perimeter?

   It is the most symmetric and is more able to enclose more rows of "area" than the longer skinny rectangles. {Algebraically, we have $A = W(16 - W)$ and the vertex is halfway (8) between the two zeroes (0 & 16).}

8. With any graph, we should ask, "is it appropriate to connect the points?" If the cases can only be whole numbers, we shouldn't, but lengths can be any value (whole or with a fractional or decimal part) and so we should connect the plotted points in a way that follows the pattern of those already graphed. That connecting line or curve means there are an infinite number of other possibilities! Connect the points in your two graphs above. If you think you are not sure how to draw the graph across the gaps, plot more points that fit the requirements of the data.

9. With any graph, we need to ask, "is it appropriate to extend the graph beyond the points?" Conencting points, as above, is called interpolation (filling in *within*). Extending a pattern is called extrapolation (extending *beyond*). In the case of extrapolation, we need to be careful not go too far (e.g., we would not want to claim that a rectangle can have sides with negative lengths). Extend the above graphs until you think they can no longer represent a rectangle. Do your graphs end? If so, where? If not, why not?

   Both graphs end when they hit an axis (one of the dimensions is 0). {These are called *degenerate* figures – they are not really rectangles, but they are the extreme case.}

10. Which rectangle has the <u>least</u> possible area for a perimeter of 32 cm? How can you convince someone that your answer is the actual minimum possible?

    If we rule out degenerate rectangles, no rectangle has the smallest possible area. We can always choose an ever-smaller width closer to 0. In that case, my graph should have had an open circle at the endpoints.

11. If you had 32 meters of fence, what rectangle would enclose the greatest amount of land? Can you think of another shape that might hold even more land for the same amount of fencing? Show some calculations to find the area for your proposed shape.

    A circle. $32 = 2\pi r$. $r = 16/\pi$. $A = \pi r^2 = \pi(16/\pi)^2 = 256/\pi \approx 81.5\ cm^2$.

12. You have a rectangle with an **<u>area</u>** of **20 cm^2**.

a. Fill the table below with many possible dimensions for your rectangle. **Don't limit yourself to whole numbers.** You can use the centimeter grid below to draw some of your rectangles.

b. Graph the values on the graph on the next page. **Label** the x-axis "width" and the y-axis "height". Note the new scale and try to use as much of the graph as possible (practice your interpolation and extrapolation skills!).

<table>
<tr><th>Width</th><th>Height</th><th>Perimeter</th><th>Area</th></tr>
<tr><td>1</td><td>20</td><td>42</td><td>20</td></tr>
<tr><td>2</td><td>10</td><td>24</td><td>20</td></tr>
<tr><td>4</td><td>5</td><td>18</td><td>20</td></tr>
<tr><td>$\sqrt{20}$</td><td>$\sqrt{20}$</td><td>$4\sqrt{20}\approx 17.88$</td><td>20</td></tr>
<tr><td>1/2</td><td>40</td><td>81</td><td>20</td></tr>
<tr><td>2.5</td><td>8</td><td>21</td><td>20</td></tr>
<tr><td>0.1</td><td>200</td><td>400.2</td><td>20</td></tr>
<tr><td colspan="3" rowspan="7">All of the above pairs flipped.</td><td>20</td></tr>
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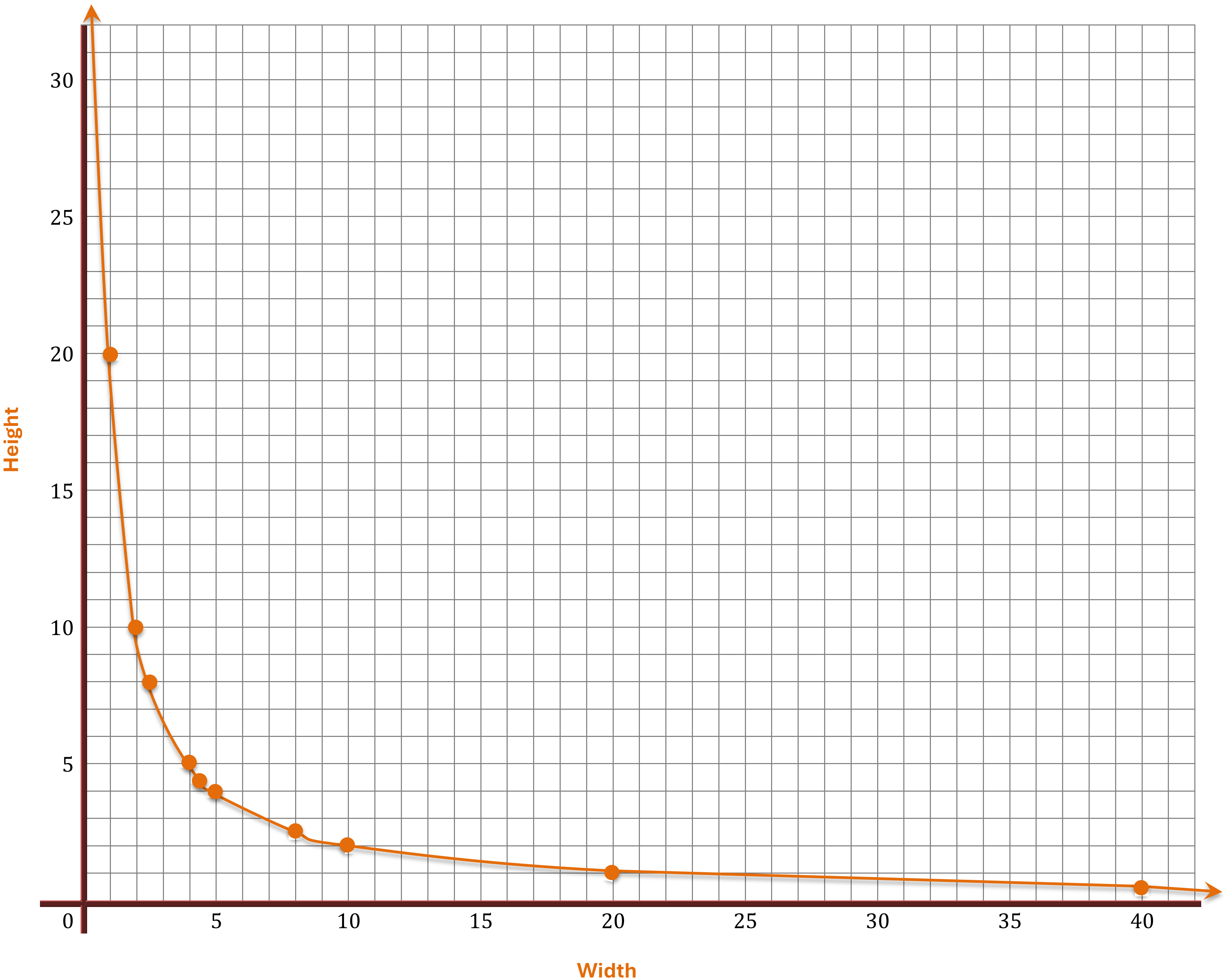
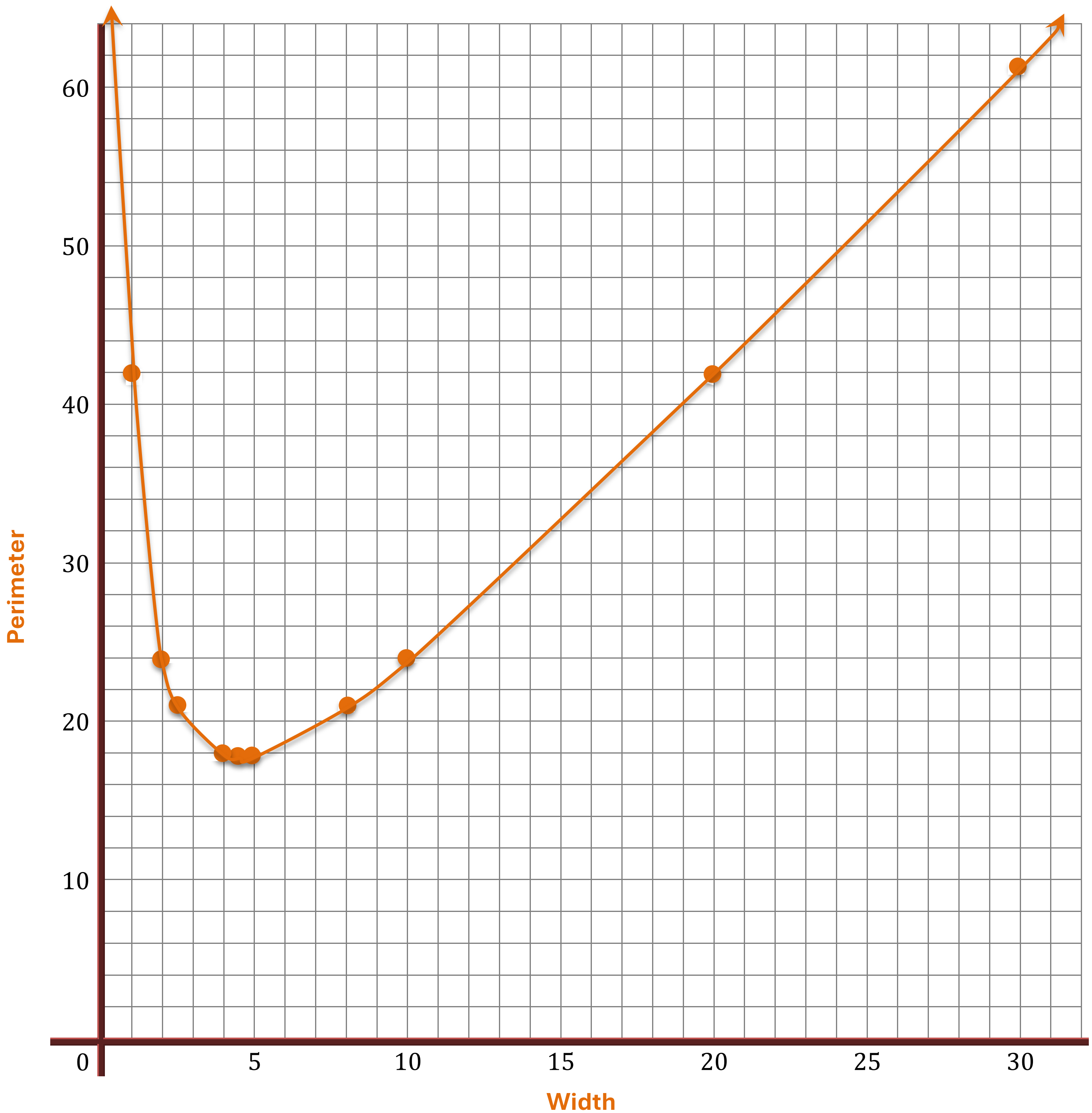


Height
30
25
20
15
10
5
0
5
10
15
20
25
30
35
40
Width

13. Return to the area chart two pages ago and complete the perimeter column for each rectangle.
14. Plot points on the graph below with your x-axis being the **width** of each rectangle and the y-axis being that rectangle's **perimeter**. Label both axes. Note that the y-axis has a different scale with each box 2 units high (but still only one unit wide).



15. Studying the graph for problem 14., determine the dimensions of the rectangles that have the **greatest** perimeter and the **smallest** perimeter for that fixed area.

   a. Is there a maximum possible perimeter for a rectangle with an area of 20 cm^2? If so, what is it? If not, why not?

The perimeter can be as big as we desire. The smaller we make one dimension, the bigger the perimeter gets.

   b. Is there a minimum possible perimeter for a rectangle with an area of 20 cm^2? Is so, what is it? If not, why not?

The smallest perimeter, which is the most efficient one, is when both sides are the square root of 20.

16. You have a rectangular box with a volume of 64 cm^3. Find some possible dimensions. Do not limit yourself to whole numbers.

Width	Depth	Height	Volume	Surface Area	
4	4	4	64		96 cm^2
1	2	32	64		198 cm^2
2	4	8	64		112 cm^2
1	1	64	64		258 cm^2
0.1	0.1	6400	64	The stick	2,560.02 cm^2
5	5	2.56	64		96 cm^2
0.01	80	80	64	The pancake	12,803.2 cm^2

a. What is the smallest possible surface area for a box with a volume of 64 cm^3? Explain.

96 cm^2, because that is the most symmetric shape.

b. What is the largest possible surface area for a box with a volume of 64 cm^3? Explain.

There is no maximum. We can just make ever bigger "pancakes."

c. What is the smallest possible surface area for a box with a volume of 1000 cm^3? Explain.

600 cm^2, because the cube root of 1000 is 10, so a 10x10x10 box would be optimal.

d. What is the smallest possible surface area for a box with a volume of 100 cm^3? Explain.

$6\left(\sqrt[3]{100}\right)^2$ cm^2, for a $\sqrt[3]{100}$x$\sqrt[3]{100}$x$\sqrt[3]{100}$ box. This simplifies to $60\sqrt[3]{10}$ cm^2.

e. Do you think any object can have a smaller surface area and enclose 64 cm^3 than the box you found for question a.?

Yes, a sphere or more compact shape than a cube (which has those inefficient pointy parts). A cylinder would be better as well.

A room is filled with cans (cylinders). They all have a volume of 600 cm^3. Find the dimensions of at least six or more of these cans. Calculate the surface area of at least three of them.

Height	Radius	Volume	Surface area
19098.6	0.1	**600**	12000
191	1	**600**	1206
47.7	2	**600**	625
21.2	3	**600**	457
7.6	5	**600**	397
1.9	10	**600**	748
0.5	20	**600**	2573
0.02	100	**600**	62844

The volume of a cylinder is $V = \pi R^2 H$. So we want $R^2 H = 600/\pi \approx 191$.

So $H = {}^{191}/_{R2}$. Pick values for R and find H.

Note: calculus shows us that the minimum surface area for any cylinder given a fixed volume is when the diameter and height are equal (it has a square orthographic projection). Setting H to 2R and solving gives a radius of $R \approx 4.57$ and $H \approx 9.14$.

Determine a useful scale for your axes and label them and then graph the height (*x*) and radius (*y*) of each on the graph paper below.

Describe the shape of the graph.

Calculate and add additional points to get a more complete picture of its shape.

Graph scale is 2x2 per box.
The graph is asymptotic to both axes. The smaller the radius, the larger the height and the other way around. It is an inverse square relationship.