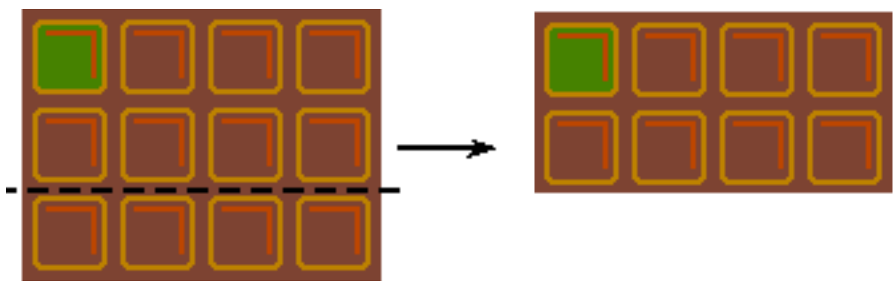
Yucky Chocolate and Chomp

The related games Yucky Chocolate and Chomp are good settings for early work with proof, particularly in class settings. These games are effective with both middle and high school classes. Both games begin with an n-by-m array of chocolate squares (n not necessarily different from m) in which the top left square of chocolate has become moldy.

Rules for the game of Yucky Chocolate

On each turn in the game of Yucky Chocolate, a player chooses to break the bar of chocolate along a **horizontal** or **vertical** line. These breaks must be between the rows of squares as shown in the figure below. The rectangle that is broken off is "eaten" by that player.



A horizontal break in the game of Yucky Chocolate leaves a 2 by 4 board

The game continues with the remaining rectangle that includes the yucky square. You can introduce this game with real chocolate, but the incentive to break off large pieces for consumption may overwhelm any other strategic thinking. Players take turns until one player, the loser, is left with just the yucky piece to eat.

Introduce your class to the rules of the game and then have them pair off to play several rounds starting with a 4 by 6 board. They can play the game on graph paper (handout appended to this guide) or with the online version at MakingMath.org, mark off the starting size of the chocolate bar, and then shade in eaten portions each turn. The game is a single HTML file. It can also be downloaded and opened in a browser on computers, ChromeBooks, or large tablets. No network is required after loading the file. After a few rounds of play, students will start to notice winning end-game strategies. In one fifth-grade class, the students observed that when a player faced a 2-by-2 board, they always lost. Given that observation, additional play led them to see why a 3-by-3 board was also a losing position. They were able to turn these conjectures into theorems with simple case-by-case analyses. For the 3-by-3 board, the symmetry of the situation meant that there were really only two distinct moves possible (leaving a 2-by-3 or 1-by-3 board). Each of these moves gave the other player a winning move (reducing the board to a 1-by-1 or 2-by-2 case).

After the class realized that the smaller square positions were losers, some students took the inductive leap to conjecture that all *n*-by-*n* boards represented losing positions. One girl, who had never studied proof by induction, excitedly began explaining how each larger square array could be turned into the next smaller one and that she could always force the game down to the proven losing square positions. She had an intuitive understanding of the validity of an inductive argument. She then stopped and realized that her opponent might not oblige her by carving off just one column or one row and that she did not know how big the next board might be. She had cast doubt on the reasoning of her own argument. She was facing another form of inductive proof in which one builds not just from the next smallest case but all smaller cases. After a while, the class was able to show that regardless of the move that an opponent facing an *n*-by-*n* board takes, there was always a symmetrical move that made a smaller square board. Therefore, they could inexorably force a win. This argument made it possible for a full analysis of the games that led to a win for the first player ($n \neq m$) and those that should always be won by the second player.

Once students have a complete understanding of Yucky Chocolate, the game provides a nice opportunity for practicing problem posing. Ask the students to each develop one or more variations of the game. What characteristics can they change? Does the game remain interesting? Does it become more complicated? Do they have to change any rules to make it still make sense? Some of the changes that students have explored include moving the location of the moldy square, making the problem three-dimensional, changing the number of players, or playing with a triangular grid of chocolate. See a version of this assignment, Even Yuckier Chocolate, at the end of this document.

Rules for the game of Chomp

The game of Chomp starts with the same slightly moldy chocolate bar, only the players take turns biting the chocolate bar with a right-angled mouth. These bites remove a chosen square and all remaining squares below and/or to the right of that square (see figure below and the articles in the bibliography for examples).

Two turns in a game of Chomp (with a weird right-angled mouth)

These bites can leave behind boards with complicated shapes that make it difficult to analyze which player should win for a given starting board. Student investigations can identify many sets of initial configurations (e.g., the 2-by-*n* or *n*-by-*n* cases) where a winning strategy can be determined and a proof produced (see Keeley And Zeilberger). Zeilberger's Three-Rowed Chomp provides an elegant existence proof that the first player in a game must always have a winning strategy. Being an existence proof, it provides no hint at how the winning strategy might be found. See also Gardner, Joyce, and Stewart for more on the game of Chomp. The article by Stewart also discusses Yucky Chocolate. The Keeley article provides a lovely discussion of one class's definitions, conjectures, and theorems about the game of Chomp.

Chomp and Yucky Chocolate Bibliography

Gardner, Martin (1986). *Knotted doughnuts and other mathematical recreations*. New York, N.Y.: W. H. Freeman and Company, 109-122.

Joyce, Helen (2001, March). *Chomp*. Available on-line at http://plus.maths.org/issue14/xfile/.

Keeley, Robert J (1986, October). *Chomp-an introduction to definitions, conjectures, and theorems*. *Mathematics Teacher*, 516-519. Available online at https://www.jstor.org/stable/27965044?seq=1.

Stewart, Ian (1998, October). *Mathematical recreations: playing with chocolate*. *Scientific American*, 122-124.

Zeilberger, Doron (2002). *Three-rowed Chomp*. Available online at http://www.math.rutgers.edu/-zeilberg/mamarim/mamarimPDF/chomp.pdf.

Yucky Chocolate **Name ____________**

Even Yuckier Chocolate

Your assignment is to modify the game Yucky Chocolate and describe the consequences. You should change just **one or two** aspects of the game. Think of all of the explicit and implicit characteristics of the game and think of ways that they might be modified.

Before you make a final decision about the design of the game for your project, play the version with a classmate a few times to make sure it's an interesting variant. You may need to try two or three different changes to find one that makes a good game.

Once you have a version that excites you, play your game several more times and try to develop winning strategies and other insights into the nature of the game. Play simple (smaller) versions and the work toward more challenging scales or configurations. Remember to look at what happens just before someone wins, and work backward from when you know you have won (or lost) to find the kinds of moves you do not want to make and the moves that put you in a good position.

Explain as best you can how to play your game well. If you say a move is "bad" or "good," then give examples that show why they benefit or disadvantage a player. Use diagrams as well as words to convey what you have discovered. As you describe your strategy, remember to explain all of the choices that are left for your opponent after you make a move. You do not have to figure out how to win your game in all circumstances, but, if an overall strategy is not discernable, try to determine optimal play for some subset of all possible games.

For your final product:

- Produce a neat and attractive version of your new game board
- Explain your rules clearly and make sure that they cover all situations that can happen in a game. Make sure they include the objective for players.
- Make sure to give your game a new name
- In a report, discuss:
    - Is your game fair? Can the first (or second) player always win?
    - What would be a good first move?
    - Are their general strategies like Yucky Chocolate's square strategy? Assume that opponents play equally well and as optimally as possible.
    - A proof of a winning strategy for any set of circumstances.